

3.3. Bounded linear operators and adjoint operators. (Continued from Lecture 22)

We introduce an important concept.

Definition. The adjoint operator T^* of an operator T in a Hilbert space H is an operator such that

$$\langle Tf, g \rangle = \langle f, T^*g \rangle$$

for all f and g in H .

We know that the adjoint of a real matrix A is the transpose A^t . This is because of the following identity:

$$\langle Ax, y \rangle = \langle x, A^t y \rangle = y^t Ax.$$

For a Hilbert-Schmidt operator, the adjoint is

$$T^*u(x) = \int_a^b k(y, x)u(y)dy.$$

The proof is as follows: we have

$$\langle Tu, v \rangle = \int_a^b (Tu)(x)v(x)dx = \int_a^b \int_a^b k(x, y)u(y)v(x)dydx.$$

We then change the order of integration to obtain

$$\langle Tu, v \rangle = \langle u, T^*v \rangle .$$

In general, we have the following theorem.

Theorem. The adjoint of a bounded linear operator of a Hilbert space always exists and is bounded linear.

Proof is omitted.

The application of the adjoint operator is given in the following theorem.

Theorem (Fredholm Alternative Theorem) Suppose L is a bounded linear operator in a Hilbert space H with closed range. Then the equation

$$Lf = g$$

has a solution if and only if

$$\langle g, v \rangle = 0$$

for all v such that $L^*v = 0$. The term “closed range” is a technical term which means that the image of L : $\{g \mid g = Lf, f \in H\}$ is a closed set in H .

The proof of the “only if” part is as follows. Suppose a solution f exists for a g , then

$$\langle g, v \rangle = \langle Lf, v \rangle = \langle f, L^*v \rangle = \langle f, 0 \rangle = 0$$

for all v such that $L^*v = 0$. The “if” part is omitted.

In terms of matrices we see that $Ax = y$ has a solution if and only if y is orthogonal to all the solutions v such that $A^t v = 0$.

Question and Answer: Is the differentiation operator a bounded linear operator?

Answer is as follows. Consider the space $C^\infty[a, b]$ consisting of all continuous functions on $[a, b]$ whose derivatives of all orders are continuous on $[a, b]$. With the maximum norm on $C^\infty[a, b]$, it becomes a normed vector space. Consider the operator D :

$$D : f(x) \longrightarrow f'(x).$$

We immediately see from the sequence

$$f_n(x) = \sin(2\pi nx), \quad f'_n(x) = 2\pi n \cos(2\pi nx)$$

that the differentiation operator cannot be a bounded operator from $C^\infty[0, 1]$ with the maximum norm to the space $C[0, 1]$, because the factor $2\pi n$ cannot be bounded for $n = 1, 2, 3, \dots$.

A summary: We generalize the equation $Ax = b$ to $L[u] = g(x)$ and show how to determine its solvability.

3.4. Spectral theory for compact operators.

(This section is important, but I failed to find time for it in class).

A bounded linear operator is called *compact* if it maps any bounded sequence $\{f_n\}_{n=1}^\infty$ into a sequence $\{Lf_n\}_{n=1}^\infty$ that has a convergent subsequence.

Any linear transformation in \mathbb{R}^n is compact.

Theorem. Any Hilbert-Schmidt operator is compact.

For a compact operator, we have a spectral theory just like the eigenvalue problem for matrices. For a square matrix A , if there exist a number λ and a nonzero

vector u such that there holds

$$Au = \lambda u$$

then λ is called an *eigenvalue* of A and u is called an associated *eigenvector*. For a large class of $n \times n$ matrices A , we know that there exist n eigenvalues and associated eigenvectors.

For the stress and strain tensors, we can study their principal axes, which are eigenvectors of the associated matrices.

For any linear operator the definition of eigenvalues and eigenvectors are the same. If there exist a number λ and a nonzero member u such that there holds

$$Lu = \lambda u$$

then λ is called an **eigenvalue** of L and u is called an associated **eigenfunction**.

For a compact linear operator L , we can prove that there exists at least one eigenvalue and an associated eigenfunction. Under general conditions, a compact linear operator L has infinitely many eigenvalues

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots,$$

with associated eigenfunctions

$$u_1, u_2, u_3, u_4, \dots.$$

These eigenfunctions form an orthogonal basis for H so that any function in H can be written as an infinite series

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \dots.$$

A bounded linear operator on H can be represented by an infinite matrix.

Example. Consider

$$\frac{d^2 u}{dx^2} + \lambda u = g(x), \quad 0 < x < 1$$

with boundary conditions

$$u(0) = u(1) = 0.$$

Any solution u will also satisfy the integral equation

$$u(x) + \lambda \int_0^1 k_0(x, y) u(y) dy = G(x)$$

where

$$G(x) = \int_0^1 k_0(x, y)g(y)dy.$$

When $g(x) = 0$, we have an eigenvalue problem

$$\int_0^1 k_0(x, y)u(y)dy = -\frac{1}{\lambda}u.$$

One can verify that we have

$$\lambda = (n\pi)^2, \quad n = 1, 2, 3, \dots ;$$

and

$$u = \sin(n\pi x).$$

The verification can be done through the differential equation rather than the integral equation.

If λ is not equal to any of the eigenvalues, then the equation

$$Lu(x) = G(x),$$

where

$$Lu(x) = u(x) + \lambda \int_0^1 k_0(x, y)u(y)dy,$$

has a unique solution for any $g(x)$. If λ is one of the eigenvalues, then $g(x)$ needs to be orthogonal to all the solutions v to

$$L^*v(x) = v(x) + \lambda \int_0^1 k_0(y, x)v(y)dy = 0.$$

We note that T is called **self-adjoint** if $T^* = T$. So the function $k_0(x, y)$ yields a self-adjoint operator. A Hilbert-Schmidt operator is self-adjoint when $k(x, y) = k(y, x)$. The eigenvalues of self-adjoint compact linear operators are all real numbers.

—End of Chapter III —