Bounded linear functionals, Riesz representation, Dirac delta

There are generally many bounded linear functionals. But there is an excellent representation of bounded linear functionals on Hilbert spaces.

**Theorem** (Riesz representation theorem) Any bounded linear functional $T$ on a Hilbert space $H$ can be represented by a member $g \in H$ in the form of the inner product

$$Tf = \langle g, f \rangle, \quad \text{for all } f \in H.$$  

**Example 3.** On $\mathbb{R}^n$, a linear function $f(x_1, x_2, \cdots, x_n)$, i.e., it satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all real $\alpha$ and $\beta$ and all points $x$ and $y$ in $\mathbb{R}^n$, takes the form of an inner product with a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$:

$$f(x_1, x_2, \cdots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha \cdot x.$$  

(Comment: What is normally called linear in $\mathbb{R}^n$:

$$f(x_1, x_2, \cdots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha \cdot x + b$$

where $b$ is a scalar number, should be, and is actually called affine.

Let $p$ be a number such that $1 < p < \infty$. And let $q$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$  

We have

**Theorem** (Riesz representation part II) Any bounded linear functional $T$ on $L^p[a, b]$ can be represented by a function $g \in L^q[a, b]$ in the form

$$Tf = \int_a^b f(x)g(x)dx.$$  

We note that the Hölder inequality is helpful:

$$|\int_a^b f(x)g(x)dx| \leq \|f\|_{L^p} \|g\|_{L^q}.$$
**Dirac delta functional.** Consider the Banach space $C[a, b]$. Its bounded linear functionals form a space called *finite Borel Measures*, which include the Dirac delta functional $\delta(x - x_0)$. Let us first consider the example functional:

$$Tf = f(x_0);$$

i.e., $T$ takes any continuous function $f(x)$ to a number $f(x_0)$ where $x_0$ is a point in the interval $(a, b)$. This functional is linear since

$$T(\alpha f + \beta h) = \alpha f(x_0) + \beta h(x_0) = \alpha T(f) + \beta T(h).$$

It is bounded since

$$|Tf| = |f(x_0)| \leq \max_{x \in [a, b]} |f(x)| = \|f\|_{C^0}.$$ 

Traditionally this function is written as

$$Tf = \int_a^b \delta(x - x_0)f(x)dx \quad (= f(x_0))$$

in line with the $L^q$ representation of functionals on $L^p[a, b]$. In this representation, the $\delta(x - x_0)$ was regarded as a generalized function with the properties:

a. $\int_a^b \delta(x - x_0)dx = 1, (x_0 \in [a, b])$

b. $\delta(x - x_0) = 0$ for $x \neq x_0$.

We note that $\delta(x - x_0)$ is not a functional on the space $L^p[a, b]$ since an $L^p$ function may not be defined on individual points.

Functionals are defined by their actions on functions. We regard a functional as well-defined if its actions on all functions of a space are defined. This still leave room for the functional itself, but we regard that room as irrelevant.

**3.3. Bounded linear operators and adjoint operators.**


Similar to the linear transformations $L$ from a Euclidean space $R^n$ to $R^n$ represented by

$$y = Ax,$$

we define a **linear operator** $L$ from a Hilbert space $H$ to $H$ to be a mapping that satisfies

$$L(\alpha f + \beta g) = \alpha Lf + \beta Lg$$
for all real numbers \( \alpha \) and \( \beta \) and all members \( f \) and \( g \) in \( H \). The linear operator is called \textbf{bounded} if there exists a constant \( C \) such that

\[ \|Lf\| \leq C\|f\| \]

for all \( f \in H \).

Let us look at an example. From the differential equation

\[ \frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1 \]

with the two-point boundary value

\[ u(0) = u(1) = 0, \]

one can obtain the solution formula

\[ u(x) = \int_0^1 k_0(x,y)f(y)dy \]

where

\[ k_0(x,y) = \begin{cases} y(x-1), & 0 \leq y < x \leq 1 \\ x(y-1), & 0 \leq x < y \leq 1. \end{cases} \]

This solution formula is a bounded linear operator for \( f(x) \in L^2[0,1] \) to \( u(x) \in L^2[0,1] \), see the next theorem.

**Theorem.** For any \( k(x,y) \) such that

\[ \int_a^b \int_a^b k^2(x,y)dxdy = C < \infty, \]

the operator

\[ Tu(x) = \int_a^b k(x,y)u(y)dy \]

is a bounded linear operator from \( L^2[a,b] \) to \( L^2[a,b] \).

This operator is called a \textbf{Hilbert-Schmidt operator}.

**Proof.** We use Cauchy-Schwarz inequality

\[
\|Tu(x)\| = (\int_a^b (Tu(x))^2 dx)^{1/2} \\
= (\int_a^b (\int_a^b k(x,y)u(y)dy)^2 dx)^{1/2} \\
\leq (\int_a^b (\int_a^b k^2(x,y)dxdy)(\int_a^b u^2(y)dy)dx)^{1/2} \\
= (\int_a^b \int_a^b k^2(x,y)dxdy)^{1/2}(\int_a^b u^2(y)dy)^{1/2} \\
= C\|u\|. \tag{1}
\]
The proof is complete.

Another example is the Fourier transform $\mathcal{F}$ that takes a function in $L^2$ to a function in $L^2$, and the operator has norm 1:

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$$

It is so amazing.

=====End of Lecture 22, Oct 23. ======