(Continued from Lecture 21) **Bounded linear functionals, Riesz representation, Dirac delta**

There are generally many bounded linear functionals. But there is an excellent representation of bounded linear functionals on Hilbert spaces.

Theorem (Riesz representation theorem) Any bounded linear functional T on a Hilbert space H can be represented by a member $g \in H$ in the form of the inner product

$$Tf \; = < g, f >, \qquad \text{for all } f \in H.$$

Example 3. On $I\!R^n$, a linear function $f(x_1, x_2, \cdots, x_n)$, i.e., it satisfies

$$f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$$

for all real α and β and all points $\mathbf{x}$ and $\mathbf{y}$ in $I\!R^n$, takes the form of an inner product with a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$:

$$f(x_1, x_2, \cdots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha \cdot x.$$

(Comment: What is normally called linear in $\mathbb{R}^n$:

$$f(x_1, x_2, \cdots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha \cdot x + b$$

where b is a scalar number, should be, and is actually called *affine*.

Let p be a number such that $1 < p < \infty$. And let q be such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We have

Theorem (Riesz representation part II) Any bounded linear functional T on $L^p[a, b]$ can be represented by a function $g \in L^q[a, b]$ in the form

$$Tf = \int_a^b f(x) g(x) dx.$$

We note that the Hölder inequality is helpful:

$$| \int_a^b f(x) g(x) dx | \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Dirac delta functional. Consider the Banach space $C[a,b]$. Its bounded linear functionals form a space called *finite Borel Measures*, which include the Dirac delta functional $\delta(x-x_0)$. Let us first consider the example functional:

$$Tf = f(x_0);$$

i.e., T takes any continuous function $f(x)$ to a number $f(x_0)$ where x_0 is a point in the interval (a,b). This functional is linear since

$$T(\alpha f+\beta h) = \alpha f(x_0) + \beta h(x_0) = \alpha T(f) + \beta T(h).$$

It is bounded since

$$|Tf| = |f(x_0)| \le \max_{x\in[a,b]} |f(x)| = \|f\|_{C^0}.$$

Traditionally this function is written as

$$Tf = \int_a^b \delta(x-x_0) f(x)dx \qquad (= f(x_0))$$

in line with the L^q representation of functionals on $L^p[a,b]$. In this representation, the $\delta(x-x_0)$ was regarded as a generalized function with the properties:

a. $\int_a^b \delta(x-x_0)dx = 1, (x_0 \in [a,b])$

b. $\delta(x-x_0) = 0$ for $x \neq x_0$.

We note that $\delta(x-x_0)$ is not a functional on the space $L^p[a,b]$ since an L^p function may not be defined on individual points.

> Functionals are defined by their actions on functions. We regard a functional as well-defined if its actions on all functions of a space are defined. This still leave room for the functional itself, but we regard that room as irrelevant.

3.3. Bounded linear operators and adjoint operators.

See Text by Keener, §3.1-2. pp.101–118.

Similar to the linear transformations L from a Euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$ represented by

$$y = Ax,$$

we define a **linear operator** L from a Hilbert space H to H to be a mapping that satisfies

$$L(\alpha f + \beta g) = \alpha Lf + \beta Lg$$

for all real numbers α and β and all members f and g in H. The linear operator is called **bounded** if there exists a constant C such that

$$\|Lf\| \leq C\|f\|$$

for all $f \in H$.

Let us look at an example. From the differential equation

$$\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1$$

with the two-point boundary value

$$u(0) = u(1) = 0,$$

one can obtain the solution formula

$$u(x) = \int_0^1 k_0(x,y)f(y)dy$$

where

$$k_0(x,y) = \begin{cases} y(x-1), & 0 \leq y < x \leq 1 \\ x(y-1), & 0 \leq x < y \leq 1. \end{cases}$$

This solution formula is a bounded linear operator for $f(x) \in L^2[0,1]$ to $u(x) \in L^2[0,1]$, see the next theorem.

Theorem. For any $k(x,y)$ such that

$$\int_a^b \int_a^b k^2(x,y)dxdy = C < \infty,$$

the operator

$$Tu(x) = \int_a^b k(x,y)u(y)dy$$

is a bounded linear operator from $L^2[a,b]$ to $L^2[a,b]$.

This operator is called a **Hilbert-Schmidt operator**.

Proof. We use Cauchy-Schwarz inequality

$$\begin{aligned} \|Tu(x)\| &= (\textstyle\int_a^b (Tu(x))^2 dx)^{1/2} \\ &= (\textstyle\int_a^b (\int_a^b k(x,y)u(y)dy)^2 dx)^{1/2} \\ &\leq (\textstyle\int_a^b (\int_a^b k^2(x,y)dy)(\int_a^b u^2(y)dy)dx)^{1/2} \\ &= (\textstyle\int_a^b \int_a^b k^2(x,y)dydx)^{1/2}(\int_a^b u^2(y)dy)^{1/2} \\ &= C\|u\|. \end{aligned} \tag{1}$$

The proof is complete.

Another example is the Fourier transform $\mathcal{F}$ that takes a function in L^2 to a function in L^2, and the operator has norm 1:

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$$

It is so amazing.

=====End of Lecture 22, Oct 23. ======