3.2. Bounded linear functionals, Riesz representation, and Dirac delta functional.

See Text by Keener, §3.1-2. pp.101–118.

Recall we have introduced the three spaces: The space $C[a,b]$ of all continuous functions on the interval $[a,b]$ with the maximum norm; the space $L^2[a,b]$ as the completion of all continuous functions on $[a,b]$ in the L^2 norm, which contains all continuous functions, all step functions, and functions like $f(x) = |x|^{-1/3}$; and the space $L^1[a,b]$.

Step functions are functions that are equal to constants on subintervals of the interval $[a,b]$. They are discontinuous. But they can be approximated by continuous functions in the L^2 norm.

A characterization of a function $f \in L^2[a,b]$ is that the L^2 norm of f is finite: $\|f\|_{L^2[a,b]} < \infty$.

For a function $f(x) = x^{-\alpha},\ x \in (0,1], f(0) = 0$ to be in $L^2[0,1]$, we need to find the L^2 norm

$$\int_0^1 f^2 \, dx = \int_0^1 x^{-2\alpha} \, dx = \frac{1}{1-2\alpha} x^{1-2\alpha}|_0^1$$

which is finite only when $\alpha < \frac{1}{2}$.

Similar to an ordinary function $y = f(x)$ that is defined on the space $\mathbb{R}^n$ and takes on values in $\mathbb{R}^1$, a **functional** is a function on a function space B that determines uniquely a number in $\mathbb{R}^1$ for each member in B.

Example 1. Let $B = L^2[0,1]$ and

$$T_1 f = \int_0^1 f(x) dx.$$

Then T_1 is a functional on B.

A functional T is **linear** if it satisfies

$$T(\alpha f + \beta h) = \alpha T(f) + \beta T(h)$$

for all real numbers α and β and all members f and h in B.

We see the above T_1 is linear, but the following S

$$Sf = \int_0^1 (f(x))^2 dx$$

is not linear.

A functional T is **bounded** if there exists a number C such that

$$|Tf| \leq C\|f\|_B$$

holds for all $f \in B$.

Example 2. Consider $B = L^2[0,1]$ and a $g(x) \in L^2[0,1]$. Define

$$T_g f = \int_0^1 g(x)f(x)dx.$$

We show that T_g is a bounded linear functional. Recall Cauchy-Schwarz inequality

$$|\int_0^1 fgdx| \leq (\int_0^1 f^2 dx)^{1/2} (\int_0^1 g^2 dx)^{1/2}.$$

So $T_g f$ is a finite number for any $f \in B$. Thus T_g is defined on B as a functional. From the inequality we see that it is also bounded with the choice $C = \|g\|_{L^2}$. The linearity is obvious.

===End of Lecture 21, Oct 21, 2002======