3.1. Banach and Hilbert spaces (Continued)

See Text by Keener, §2.1. pp.59–65.

Let S be a collection of objects: $S = \{$ members $f, g, h, \cdots\}$. Suppose S has two operations, one is called addition $f+g$ and the other is called a scalar multiplication αf, where α is a real number, so that S is a vector space. Let S has a norm $\|\cdot\|$.

Definition 3.1 (Limit). A sequence $\{f_n\}_{n=1}^{\infty}$ in a normed vector space S is said to have a limit f in S (or to converge to f in S), denoted

$$\lim_{n\to\infty} f_n = f,$$

if there is such an f in S and for any $\epsilon > 0$, there exists an integer N (depends on ϵ) such that

$$\|f_n - f\| < \epsilon$$

for all $n > N$.

A concept of closeness that does not require knowledge of the limiting member is as follows

Definition 3.2 (Cauchy sequence). A sequence $\{f_n\}_{n=1}^{\infty}$ in a normed vector space S is called a *Cauchy sequence* if for any $\epsilon > 0$, there exists an integer N (depends on ϵ) such that

$$\|f_n - f_m\| < \epsilon$$

for all $n > N, m > N$.

Property: Every convergent sequence is a Cauchy sequence. This can be proved easily, which we do not have time to do, but the main step is

$$\|f_n - f_m\| = \|f_n - f + f - f_m\| \le \|f_n - f\| + \|f - f_m\|.$$

Remark: Not every Cauchy sequence converges to a limit in a normed vector space.

Definition 3.3 (Completeness). A normed vector space S is said to be complete if every Cauchy sequence converges to a limit in S.

Definition 3.4 (Banach space). A complete normed vector space is called a Banach space.

Theorem. The space $C[a,b]$ of all continuous functions on $[a,b]$ with the sup-norm is a Banach space.

We do not prove the theorem here.

Another example is that the set of all real numbers, denoted $\mathbb{R}$, with the usual addition and scalar multiplication and the absolute value as the norm is a Banach space.

But we do wonder: If S is complete with respect to one norm, is it complete with respect to other norms?

First why do we care about multiple norms? Let us look at an example. Let $v(y) = \sin y,\ 0 < y < \pi$. It is a continuous function. We can calculate its maximum norm to be 1. We can find its L^1 norm:

$$\|v\|_{L^1} = \int_0^\pi |v(y)|\, dy = 2.$$

We can find its L^2 norm squared:

$$\|v\|_{L^2}^2 = \int_0^\pi |v(y)|^2\, dy.$$

So what do they mean? First if we think v as velocity and y as time, then the maximum norm is the maximum velocity and the police might want to use it for speeding ticket. The L^1 norm is the total distance traveled, which a rental company might want to know. Second if we think v as velocity again but regard y as space variable x of a uniform materail bar with mass density m, then the kinetic energy is

$$\frac{1}{2} m \int_0^\pi v(x)^2\, dx.$$

Thus different norms of the same mathematical function may all have significance.

Let us look at the set of all continuous functions C. We wonder whether the set C is complete under the L^2 norm.

It turns out that it is not complete under the L^2 norm. For the L^2 norm, there are many holes between any pair of continuous functions. We collection all the "hole" functions, as we collection $\sqrt{2}, \sqrt{5}$, etc., to form a larger space, which is called the $L^2[a,b]$ space.

Definition 3.5. The collection of all possible functions that are limits of continuous functions on $[a,b]$ in the L^2 norm is called the space $L^2[a,b]$.

It turns out that $L^2[a,b]$ is complete with respect to its norm L^2. It is a Banach space. By the next definition, it is also a Hilbert space.

Definition 3.6 (Hilbert space). A Banach space whose norm is induced by an inner product is called a Hilbert space.

The ordinary Euclidean space R^n with the square-root norm is a Hilbert space.

The space of all possible functions that are limits of continuous functions on $[a, b]$ in the L^p norm $(1 \leq p < \infty)$:

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

is called the space $L^p[a, b]$. It is complete and is a Banach space. Except for $p = 2$, none of the other $L^p[a, b]$ spaces is a Hilbert space.

Example. The function

$$g(x) = \frac{1}{|x|^{\frac{1}{3}}} \in L^2[-1, 1]$$

which is not only discontinuous, but also unbounded.

To get to know these L^p spaces well, one needs to take the course "Real Analysis."

"Holes" between continuous functions. Let us consider a sequence of continuous functions:

$$f_n(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}(t - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} + \frac{1}{n} \\ 1, & \frac{1}{2} + \frac{1}{n} < t \le 1. \end{cases}$$

From its graph (see text book) it is clear that it "converges" to a function

$$f_0(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ \frac{1}{2}, & t = \frac{1}{2} \\ 1, & \frac{1}{2} < t \le 1. \end{cases}$$

We show this convergence is not in the sup-norm:

$$\|f_n - f_0\| = \sup_{t \in [0,1]} |f_n(t) - f_0(t)| = 1/2,$$

which does not converge to zero. However, we can show that it converges to f_0 in the L^2 norm as follows. We first have

$$\begin{aligned} \|f_n - f_0\|_{L^2} &= \left(\int_0^1 (f_n(t) - f_0(t))^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2} + \frac{1}{n}} (f_n(t) - f_0(t))^2 dt \right)^{\frac{1}{2}}, \end{aligned} \tag{1}$$

where we have shortened the interval of integration since the difference between $f_n(t)$ and $f_0(t)$ is zero outside the middle interval. Note that the middle portion shrinks in length to zero while the integrand is no larger than 1, we have

$$\begin{aligned} \|f_n - f_0\|_{L^2} &\le \left(\int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2} + \frac{1}{n}} (1)^2 dt \right)^{\frac{1}{2}} \\ &= \left((\tfrac{1}{2} + \tfrac{1}{n}) - (\tfrac{1}{2} - \tfrac{1}{n}) \right)^{\frac{1}{2}} \\ &= \left(\tfrac{2}{n} \right)^{\frac{1}{2}} \end{aligned} \tag{2}$$

which goes to zero. This shows that f_0 is a limit in the L^2 norm. But f_0 does not belong to C. This is rather like the limit $\sqrt{2}$ of rational numbers. When we collect all possible functions that are limits of continuous functions on $[a, b]$ in the L^2 norm, we get a larger class of functions than $C[a, b]$.

====End of Lecture 20, Oct 18=====