

Chapter III. Applied Functional Analysis

Course contents outline:

3.1. Normed vector (linear) spaces of functions, Cauchy sequence, completeness, Banach and Hilbert spaces;

3.2. Bounded linear functional and operator; Riesz Representation theorem, adjoint operator,

3.3. Fredholm alternative theorem;

3.4. Spectral theory for compact operator.

3.1. Banach and Hilbert spaces

See Text by Keener, §2.1. pp.59–65.

Motivation. When we solve algebraic equations such as

$$x^2 - 2 = 0,$$

we need numbers, really we need not only rational numbers, but all real numbers and even complex numbers. When we solve differential equations we will need functions. So we now study sets of functions.

Consider the set C of all real continuous functions on a closed interval $[a, b]$. We know that the sum of two such continuous functions is still a continuous function, and a scalar number multiplying a continuous function yields still a continuous function. As a matter of fact, these two operations satisfy all the eight properties in the definition of a *vector (linear) space*, see p.1 of text book by Keener. Thus C with these operations is a vector space.

Additionally, the operation

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

qualifies as an *inner product*, where f and g are two members of C . We recall the four defining properties of an inner product:

1. $\langle f, f \rangle \geq 0$ for all f , $\langle f, f \rangle = 0$ only when $f = 0$;
 2. $\langle f, g \rangle = \langle g, f \rangle$;
 3. $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$;
 4. $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$
- (1)

for all members of C and all real numbers α .

In applications we realize that one important concept is the distance

$$d(f, g)$$

between two members f and g . In a vector (linear) space, the distance is the same as the magnitude of the difference:

$$d(f, g) = d(f - g, 0).$$

This is similar to the vector space of the Euclidean space in the plane. Thus the magnitude or norm of a vector is important.

We use the following

$$\|f\|_{C^0} = \max_{x \in [a, b]} |f(x)|$$

as a measure of magnitude for a member f of C . Here we use the symbol \in to represent “belong to”. This magnitude will be referred to as the *maximum norm*, *super norm*, or *uniform norm*. To qualify for a *norm*, as it is widely known in this field, it needs to satisfy three properties

1. $\|f\| > 0$ for all $f \neq 0$, $\|f\| = 0$ when $f = 0$;
 2. $\|\alpha f\| = |\alpha| \|f\|$;
 3. $\|f + g\| \leq \|f\| + \|g\|$ (triangle inequality)
- (2)

for all members f, g , and all real α . We can prove that $\|f\|_{C^0}$ satisfies these properties and thus is a norm. So far the set C is a *normed vector space*.

The inner product induces a natural norm

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

It is called the L^2 norm, denoted by $\|f\|_{L^2}$. We can prove that it satisfies the three norm properties and thus is a norm.

Other normed spaces that we know are the Euclidean spaces R^n with norm

$$\|x\| = \left(\sum_{i=1}^n (x_i)^2 \right)^{1/2}$$

for $x = (x_1, x_2, \dots, x_n)$. We note that the same set of vectors can have different norms. For example we can introduce

$$\|x\|_{\max} = \max_{1 \leq i \leq n} |x_i|$$

as a norm for R^n . It can be verified that it is indeed a norm. There are other norms. A metaphor is that I can use the height measurement to select a student from the class for an award; I can use the weight measurement for selection; or I can use the Final Exam score for selection. Depending on the composition of the class, the awardee may be the same or different for different selection principles. For a given composition, two selection principles are called equivalent if the awardee is the same. The two norms of R^n mentioned above are equivalent since they yield the same geometry/topology. (Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called equivalent if there are two constants c and C such that

$$c\|f\|_1 \leq \|f\|_2 \leq C\|f\|_1$$

for all f in the set. A consequence of the equivalence of norms is that the completions of the space under two equivalent norms are the same.) But the two norms for C are not equivalent as we will see shortly.

Completeness. Consider a sequence of rational numbers

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad \dots$$

This sequence converges to $\sqrt{2}$, which is not a rational number. It means that there are a lot of “holes” in the rational number system. Any convergent set of real numbers

$$x_1, x_2, x_3, \dots,$$

however, converges to a real number. This means that the real number system is complete. We will be specific later about completeness. Completeness is important in approximation theory.

So, is the normed space C complete? The answer is not a simple yes or no, it is “it depends”.

Q: How many corners are left if one corner is cut away from a rectangular table?

====End of Lecture 19, Wed Oct 16.=====