**2.2.4. Taylor series.**

We would like to expand an analytic function in a powerful series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

and $z_0$ is a convenient point for an application. Using Cauchy integral formula for derivatives (Corollary 2.4), we find that

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

for any simple contour $C$ that contains $z_0$.

We can use other simple ways to find Taylor series: For example we have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

valid for all $|z| < 1$.

Another example is

$$\frac{z}{1-z} = z + z^2 + z^3 + \cdots.$$

A third example is

$$\frac{1+z}{1-z} = \frac{1}{1-z} + \frac{z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \cdots.$$

Using these lecture notes together with the text book is recommended.

**2.3. Application: Inviscid incompressible steady potential flow.**

Text: pp.228–230.

Take a two-dimensional inviscid fluid of density $\rho$ and velocity vector $\mathbf{u} = (u, v)$. Suppose there is no source and sink, then

$$\int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} \, d\ell = 0$$

for any domain $\Omega$ in the fluid, where $\mathbf{n}$ is the unit exterior normal to the boundary and $d\ell$ represents length element. Using the divergence theorem we find

$$\int_{\partial\Omega} \rho \mathbf{u} \cdot \mathbf{n} \, d\ell = \int_{\Omega} \nabla \cdot (\rho \mathbf{u}) \, dxdy = 0.$$

Since the domain is arbitrary, we conclude that $\nabla \cdot (\rho \mathbf{u}) = 0$. Assuming that the fluid is incompressible; i.e., $\rho =$ constant, then

$$\nabla \cdot \mathbf{u} = 0.$$

If the velocity $\mathbf{u}$ is a gradient of a scalar function $\phi(x, y)$:

$$\mathbf{u} = \nabla \phi,$$

then the flow is called a *potential flow*. A potential flow is also called irrotational or curl free flow since one can readily check that the curl of $\mathbf{u}$ is zero. In 2-dimensions, the curl is defined as $\text{curl}(u, v) = \partial_x v - \partial_y u$. The scalar function $\phi(x, y)$ is called a potential function (or simply potential) of the flow.

For incompressible and potential flow we have

$$\nabla \cdot \mathbf{u} = \frac{\partial^2 \phi}{\partial^2 x} + \frac{\partial^2 \phi}{\partial^2 y} \equiv \Delta \phi = 0.$$

Thus the potential (function) $\phi$ of a potential flow satisfies the Laplace equation for which we have already a solution formula from last lecture.

Another physical quantity that satisfies the Laplace equation $\Delta \phi = 0$ is temperature $\phi$ in a thermal equilibrium.

If we let $f(z) = \phi(x, y) + i\psi(x, y)$ be an analytic function, then both $\phi$ and $\psi$ satisfy the Laplace equation. Also, if we have a $\phi$ that satisfies the Laplace equation, we can always find a $\psi$, for example by

$$\psi(x, y) = \int_C \partial_x \phi \, dy - \partial_y \phi \, dx$$

such that the $f(z) = \phi(x, y) + i\psi(x, y)$ is analytic. Here $C$ can be any path from the origin to the point $(x, y)$. We can use Green's formula and Cauchy-Riemann conditions to verify those.

The physical interpretation of $\psi$ is very interesting. We can verify that

$$\nabla \phi \cdot \nabla \psi = 0$$

since $\phi_x = \psi_y, \psi_x = -\phi_y$. Thus the level curves of $\phi$ and $\psi$ are orthogonal. Hence the velocity $\mathbf{u}$ points along the tangent of the level curves of $\psi$. Thus the fluid flows along the level curves of $\psi$, which are called *streamlines*.

An example of such a flow is provided by such a function

$$f(z) = (z + \frac{a^2}{z})V,$$

where $a > 0$ and V are constants. This $f(z)$ is analytic away from $z = 0$. The real part

$$\phi = (x + \frac{a^2 x}{x^2 + y^2})V$$

is harmonic, whose gradient

$$\mathbf{u} = \nabla\phi$$

is the velocity field for the flow past a (cross section of) circular cylinder at $|z| = a$. The imaginary part of $f(z)$ is

$$\psi = (y - \frac{a^2 y}{x^2 + y^2})V,$$

whose level curves $\psi$ =constant are streamlines which are the dominating lines in Figure 1.

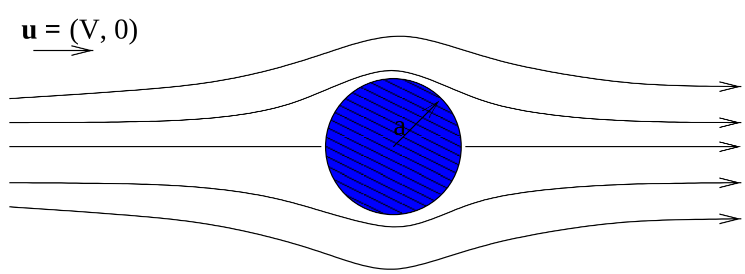


Figure 1. Flow past a cylinder.

Other notes: 1. There is a correction to the solution of HW No. 5, problem 5, page 5. Please down-load a new copy.

2. There is some addition to Lecture 16. Please down load a new version.

3. No new homework assignment this week. HW no. 6 due Wed Oct 16.

4. Exam is this Friday. Mock exam is uploaded. Its answer has a new version.

5. Feedback concerns addressed. There will be more examples in the lecture notes. Applications will be mentioned as much as possible.

===End of Lecture 18, Oct 9 =====