2.2.3. Cauchy integral formula

We have found that contour integrals of analytic functions are always zero. Only a few integrands with singularities result in nonzero values. The following Cauchy integral formula describes contour integrals extremely well.

Theorem 2.3. (Cauchy integral formula) Let C be a simple non-self-intersecting closed curve traversed counterclockwise. Suppose $f(z)$ is analytic everywhere inside C. For any point z inside C, there holds

$$\int_C \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z). \tag{1}$$

Proof. For any $\epsilon > 0$ fixed, we deform the curve C to C' where C' is the circle $|\xi - z| = \epsilon$ such that $|f(z) - f(\xi)| < \epsilon$ for all points ξ inside C'. Note that the integrand in (1) $f(z)/(\xi - z)$ is analytic in the region between C and C', we conclude that the integral in (1) is equal to the same integral over C'. (This can be achieved by the previous Theorem and a double-sided cut (or bridge) connecting C and C'.) Now on C' we have

$$\begin{aligned} \int_C \tfrac{f(\xi)}{\xi-z} d\xi &= \int_{C'} \tfrac{f(\xi)}{\xi-z} d\xi \\ &= f(z) \int_{C'} \tfrac{1}{\xi-z} d\xi + \int_{C'} \tfrac{f(\xi)-f(z)}{\xi-z} d\xi \\ &= 2\pi i f(z) + i \int_0^{2\pi} [f(z + \epsilon e^{i\theta}) - f(z)] d\theta \\ &= 2\pi i f(z) + iI \end{aligned} \tag{2}$$

where the integral I is such that $|I| \leq 2\pi\epsilon$. Let $\epsilon \to 0$. We recover the Cauchy integral formula. This completes the proof of the theorem.

Corollary 2.4. Under the same assumptions of theorem 2.3, there hold

$$\int_C \frac{f(\xi)}{(\xi - z)^2} d\xi = 2\pi i f'(z) \tag{3}$$

and

$$n! \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi = 2\pi i f^{(n)}(z) \tag{4}$$

for all n-th (n a positive integer) order derivatives. And thus analyticity implies that $f(z)$ is infinitely differentiable.

Corollary 2.5. (Poisson formula) A solution to the boundary value problem of the Laplacian

$$\begin{cases} \Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{in } x^2 + y^2 \leq 1 \\ u(r, \theta) = u_0(\theta) \quad \text{on the boundary r} = 1 \end{cases} \tag{5}$$

where (r,θ) is the polar coordinate and $u_0(\theta)$ is a given continuous function, is given by the formula

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u_0(\theta)\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}d\theta.$$

Proof. Consider an analytic function $f(z) = u(x,y) + iv(x,y)$ in $r < 1$. We have the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

So we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}),$$

thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

That is, the real part of an analytic function is a harmonic function (satisfying the Laplace equation). Now we use the Cauchy integral formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z}d\theta \quad (\text{letting } \xi = e^{i\theta})$$

for z inside the unit circle and the same formula

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\xi)\xi}{\xi - (\bar{z})^{-1}}d\theta$$

applied at the point $(\bar{z})^{-1}$ which is outside of the unit circle ($|\frac{1}{\bar{z}}| > 1$ if $|z| < 1$.) Noting that $\xi = (\bar{\xi})^{-1}$ on the unit circle, we can add the previous formulas

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)[\frac{\xi}{\xi - z} - \frac{1/\bar{\xi}}{1/\bar{\xi} - 1/\bar{z}}]d\theta.$$

Or

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{1-|z|^2}{|\xi - z|^2}d\theta.$$

Taking the real part of the formula, we obtain Poisson formula. This completes the proof.