Chapter II. Complex Variables

Dates: October 2, 4, 7, 2002.

These three lectures will cover the following sections of the text book by Keener.

§6.1. Complex valued functions and branch cuts;

§6.2.1. Differentiation and analytic functions, Cauchy-Riemann conditions;

§6.2.2. Integration;

§6.2.3. Cauchy integral formula;

§6.2.4. Taylor series and expansion.

2.1. Complex valued functions.

1. Complex numbers. We introduce the imaginary number i, whose square is -1: $i^2 = -1$. Complex numbers are in the form $a + ib$ where a and b are real numbers. Complex numbers can be represented in the Argand diagram by the vector (a, b): (Figure to be provided later). Addition and subtraction of two complex numbers are simple:

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i.$$

Multiplication and division are as follows:

$$\begin{aligned} (a+bi)(c+di) &= (ac - bd) + i(ad + bc) \\ \tfrac{a+bi}{c+di} &= \tfrac{(a+bi)(c-di)}{c^2+d^2} \end{aligned} \tag{1}$$

provided that $c^2 + d^2 \neq 0$ for the division. From these one can calculate the power $(a + bi)^n$ when n is an integer.

2. Functions. Let $z = a + bi$. We call z a complex variable when we use z as a variable. In general we let $z = x + iy$ to be consistent with our habit of real variables. Consider

$$f(z) = z^2.$$

It is called a complex valued function. Other complex valued functions are z^3,

$$g(z) = \frac{z+1}{z-1}; \quad h(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers. An important function is

$$f(z) = \bar{z} = x - iy$$

where the bar is called "complex conjugate."

We introduce the exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for all complex z. Note that this definition is consistent with the real exponential. We see this opens a new world. First we see

$$\begin{aligned} e^{i\theta} &= \textstyle\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} &&= \textstyle\sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \\ & &&= \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\ & &&= \cos\theta + i \sin\theta. \end{aligned} \quad (2)$$

Multiplying it with any real number r, we find

$$re^{i\theta} = r\cos\theta + ir\sin\theta.$$

If r and θ is the polar representation of the point (a, b), then we find the *polar representation* of complex numbers:

$$a + bi = re^{i\theta}.$$

Here there is no restriction on θ. From this we have a special case:

$$e^{i\pi} + 1 = 0$$

which is called *Euler's identity.* This identity is fun to watch since it involves the simplest symbols of mathematics: $0, 1, +, =, i$ and two transcendental numbers e and π. Another version is

$$e^{-i\pi} + 1 = 0$$

which involves additionally the minus $-$ sign. In polar representation, multiplications of complex numbers is extremely easy:

$$(a + bi)(c + di) = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1+\theta_2)}.$$

We have more examples:

$$\begin{aligned} e^z &= e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \\ e^{z_1+z_2} &= e^{z_1} e^{z_2}. \end{aligned} \quad (3)$$

3. Inverse functions of z^2 and e^z

We know that $\sqrt{5}$ is a number 2.236... and satisfies the equation $x^2 = 5$. Another solution to this equation is $-\sqrt{5}$. We can verify that both

$$z_1 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \quad z_2 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

satisfy $z^2 = i$. So there are two answers z_1, z_2 to the square root equation $z^2 = i$. Thus in general there are multiple solutions to the square root. Consider the equation $w^2 = z$. We calculate a solution

$$w_1 = \sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}.$$

We can verify that this satisfies $w^2 = z$. When we restrict θ to be in $[0, 2\pi)$, this root is called the *principal branch*. We can see another solution is

$$w_2 = \sqrt{z} = \sqrt{re^{i(\theta+2\pi)}} = \sqrt{r}e^{i(\theta/2+\pi)} = -\sqrt{r}e^{i\theta/2}.$$

If you try all possible representations $\theta + 2n\pi$ where n is an integer, you only find the previous two roots and no more.

For the exponential function $w = e^z$ we define the inverse $w = \ln z$ as

$$\ln z = w \quad \text{iff} \quad z = e^w.$$

One solution is

$$\ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta} = \ln r + i\theta \qquad \text{for } z = re^{i\theta}.$$

Or simply

$$\ln z = \ln r + i\theta \qquad \text{for } z = re^{i\theta}.$$

We note that if a $\ln z$ works, then $\ln z + 2n\pi i$ all work for all integer n:

$$e^{\ln z + 2n\pi i} = e^{\ln z}e^{2n\pi i} = e^{\ln z} = z.$$

So there are multiple inverses. The branch with the restriction

$$-\pi < \text{ Im } (\ln z) \leq \pi$$

is called the *principal branch*. The ray $\theta = \pi$ is called a *branch cut*. The origin is called a *branch point*. Any continuous interval of 2π length is called a branch. The principal branch for a function may vary from discipline to discipline.

==End of Lecture 15=== (Excluding lecture on “memorizing formula/knowledge”)