1.10. Second-order Tensors. (Cont'd)

1.10.1. The Stress Tensor.

Consider a continuous medium of material, such as an elastic medium (e.g., rubber), a rigid body, or fluid (air or water) . Use a surface to separate the medium

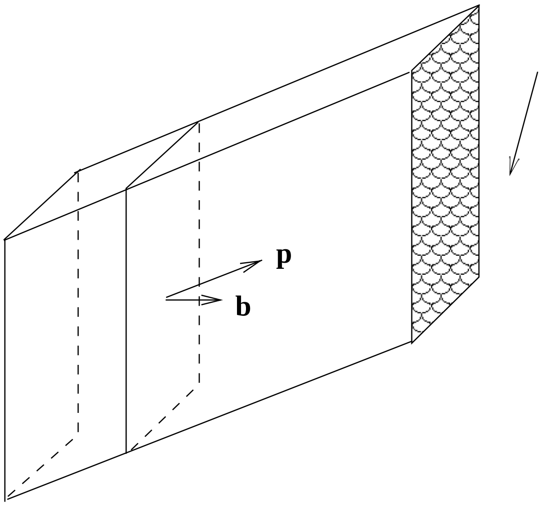


Figure 1.10.0 The left part is supporting the right part with stress {boldface} p through the crosssection with normal {boldface} b.

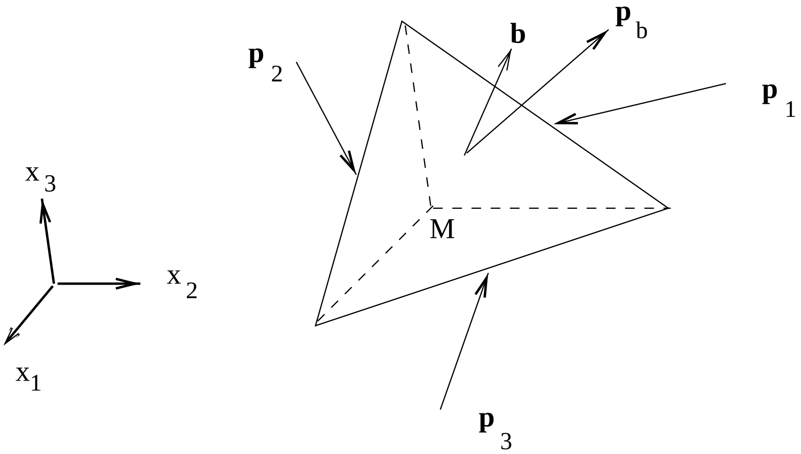


Figure 1.10.1. Tetrahedron at M with stress vectors.

you will encouter a force acting between them. See Figure 1.10.0.

The total force divided by the total surface area is called the stress (vector). The stress depends on the location in the medium and the normal direction of the surface. It is possible that we can factor out the direction part of the stress vector to form a quantity called the stress tensor so that the stress vector depends bilinearly on the tensor and the direction of the surface.

Take a rectangular coordinate system K. Take an arbitrary point M in the elastic medium. Take a tetrahedron with M being one vertex, so that the three faces passing through M are parallel to the coordinate surfaces, see Figure 1.10.1.

Let $\mathbf{b}$ be the exterior normal to the slant surface, with area $d\sigma_b$. Let $\mathbf{p}_b$ be the stress (force/unit area) onto the exterior of the tetrahedron through the slant surface. Let $\mathbf{p}_i$ be the stress onto the tetrahedron through the surface that is perpendicular to the i-th axis. Let $\mathbf{a}$ be the acceleration of the tetrahedron and $\mathbf{f}$ be the body force per unit mass. By Newton's second law, we have

$$\mathbf{a}dm = \mathbf{f}dm - \mathbf{p}_b d\sigma_b + \Sigma_i \mathbf{p}_i d\sigma_i.$$

Let dm go to zero, and note that volume goes to zero faster than corresponding surface area, we find

$$\mathbf{p}_b d\sigma_b = \mathbf{p}_i d\sigma_i.$$

Note the area formula

$$d\sigma_i = d\sigma_b \cos(\mathbf{b}, \mathbf{i}_i) = b_i d\sigma_b.$$

We find a very interesting equation

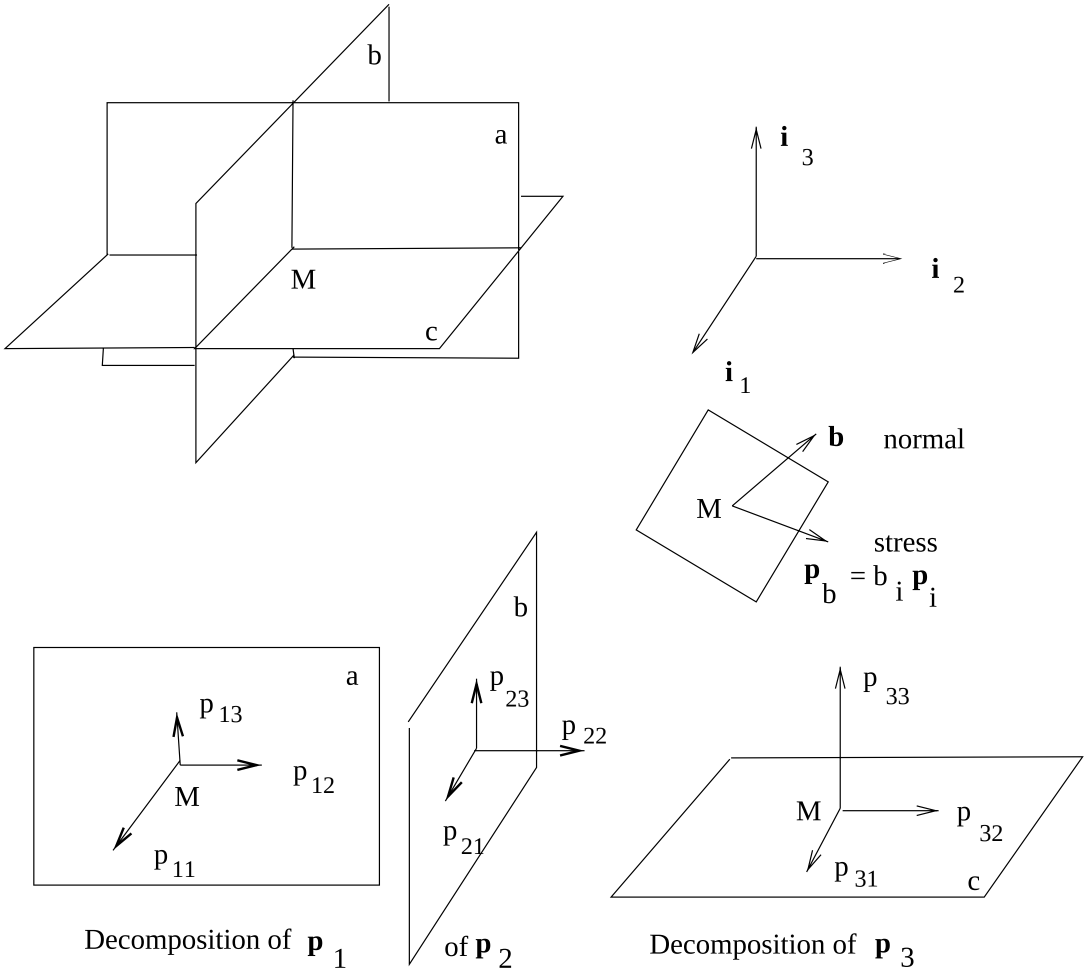
$$\mathbf{p}_b = \mathbf{p}_i b_i.$$

Projecting to $\mathbf{i}_k$ we find

$$p_{bk} = p_{ik} b_i.$$

Definition (*Stress tensor*). The (p_{ik}) is called the stress tensor. Normal stresses are p_{11}, p_{22}, p_{33}. Tangential (shearing) stresses are $p_{ij} (i \neq j)$.

We see that p_{ik} is the k-th component of the force per unit area on the surface whose normal is $\mathbf{i}_i$. See Figure 1.10.2.



Decomposition of **p** 1

of **p** 2

Decomposition of **p** 3

Figure 1.10.2. Components of the stress tensor.

Note that **b** is arbitrary since the tetrahedron is not necessarily regular. We have

$$\mathbf{p}_b = p_{bk}\mathbf{i}_k = p_{ik}b_i\mathbf{i}_k,$$

which determines the stress (vector) on all surfaces.

Note now that (p_{ik}) depends only on the point M, not **b**.

Real Meaning of Stress Tensor. Once **b** is specified, the stress tensor (p_{ik})

and **b** give the stress

$$\mathbf{p}_b = p_{ik} b_i \mathbf{i}_k \qquad (\text{ force/unit area }).$$

Once the area is specified as $d\sigma_b$, the force is

$$\mathbf{p}_b d\sigma_b.$$

A second-order (stress) tensor takes a vector (unit normal) to a (stress) vector.

Assuming that there is no internal torque, we obtain that

$$p_{ij} = p_{ji}.$$

That is, the stress tensor is symmetric. For a proof, consider the cube in Figure 1.10.3. Take the x_3 as a rotation axis, the forces that can produce torques are p_{12} and p_{21}. They have to be equal.

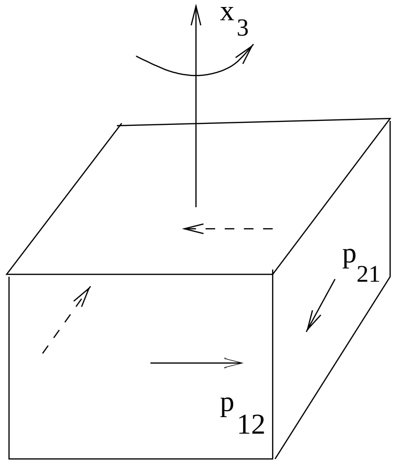


Figure 1.10.3 The stress tensor is symmetric.

It only remains to verify that p_{ik} is indeed a second-order tensor. For mathematical rigor as well as the whole point of the concept of tensor, we should verify that the stress satisfies the law of coordinate transformation. However, it is a technical point, which I choose to skip in class.

Verification of the tensor character of the stress. Since the definition of (p_{ik}) involves no restriction on the normal $\mathbf{b}$, we can take $\mathbf{b}$ to be the i-th base vector of the new coordinate system K', so that

$$\mathbf{b} = \mathbf{i}'_i$$

(K and K' have orthonomal bases $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ and $\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3$, respectively). Then projecting $\mathbf{b}$ onto the l-th axis of K gives

$$b_l = \mathbf{b} \cdot \mathbf{i}_l = \mathbf{i}'_i \cdot \mathbf{i}_l = \alpha_{i'l},$$

where $\alpha_{i'l}$ is the cosine of the angle between the i-th axis of K' and the l-th axis of K, and hence

$$\mathbf{p}_b \equiv \mathbf{p}'_i = \mathbf{p}_l b_l = \alpha_{i'l}\mathbf{p}_l = \alpha_{i'l}\mathbf{i}_m p_{lm}.$$

Finally, projecting $\mathbf{p}'_i$ onto the k-th axis of K', we obtain

$$\mathbf{p}'_i \cdot \mathbf{i}'_k = \alpha_{i'l}(\mathbf{i}_m \cdot \mathbf{i}'_k)p_{lm}$$

or

$$p'_{ik} = \alpha_{i'l}\alpha_{k'm}p_{lm}.$$

By definition, we find that (p_{ik}) transforms like a second-order tensor.

====End of Lecture 9-=====