1.8. Zeroth-Order Tensors (Scalars).

A scalar is a single function (i.e., one component) which is invariant under changes of the coordinate systems.

We deal with rectangular coordinate systems only. Thus our tensors are also called cartesian tensors.

Recall the definition of a function. A function f is a rule that maps a point of a domain Ω to a point in another domain (called range). These two functions are the same: $f(x) = e^{-x}$ for all real x and $g(z) = e^{-z}$ for all real z.

Let φ be a function of points in a domain in space. Think of φ as a physical or geometrical quantity. This function exists irrelevant of a coordinate system (e.g. temperature, density, or pressure). Suppose we have two rectangular coordinate systems K and K'. In K we have the representation $\varphi(x_1, x_2, x_3)$ of the function; while in K' we have the representation $\varphi'(x'_1, x'_2, x'_3)$ where x_i and x'_i are the coordinates of one and the same point in K and K'. If the function is a scalar, then

$$\varphi(x_1, x_2, x_3) = \varphi'(x'_1, x'_2, x'_3) \tag{1}$$

for all points in the domain. That is, let M be a geometric point (physical location), let its coordinates be (x_1, x_2, x_3) in K and (x'_1, x'_2, x'_3) in K', let the rules to determine temperature at M be φ in K and φ' in K', then the two rules φ and φ' yield the same value.

When thinking about the tensors, it is helpful to separate nature (material world) from the coordinate systems. Imagine the coordinate system is the chart on the mirror of a telescope: it moves when you move your telescope, you can attach it to a nature spot of your interest, you can rotate the chart if you like.

Example 1.8.1a. We show that the distance between two points is a scalar.

(Distance is a function of two points. We can choose to fix one point and regard the distance as a function of the other point. Thus this example fits into the equation (1).)

Proof. Let A and B be two points. Let K and K' be two rectangular coordinate systems. In these systems both A and B have coordinates:

$$\begin{aligned} &A \text{ has coordinates } x_i^A \text{ in } K, \quad \text{and } {x'_i}^A \text{ in } K'; \\ &B \text{ has coordinates } x_i^B \text{ in } K, \quad \text{and } {x'_i}^B \text{ in } K'. \end{aligned}$$

Let

$$\Delta x_i = x_i^B - x_i^A, \qquad \Delta x'_i = x'^B_i - x'^A_i.$$

Let the transformation from K to K' be

$$x'_i = \alpha_{i'k} x_k + x_{0i}.$$

Then

$$\begin{aligned} \Delta x'_i &= x'^B_i - x'^A_i &&= \alpha_{i'k} x_k^B + x_{0i} - \alpha_{i'k} x_k^A - x_{0i} \\ & &&= \alpha_{i'k}(x_k^B - x_k^A) = \alpha_{i'k} \Delta x_k. \end{aligned}$$

Thus

$$\Delta x'_i = \alpha_{i'k} \Delta x_k. \tag{2}$$

Recall Pythagorean Theorem for distance

$$(\Delta s')^2 = \sum_{i=1}^{3} (\Delta x'_i)^2.$$

Then

$$\begin{aligned} (\Delta s')^2 &= \alpha_{i'k} \Delta x_k \alpha_{i'l} \Delta x_l &&= \alpha_{i'k} \alpha_{i'l} \Delta x_k \Delta x_l \\ & &&= \delta_{kl} \Delta x_k \Delta x_l \\ & &&= \textstyle\sum_{k=1}^{3} (\Delta x_k)^2. \end{aligned}$$

Thus

$$\Delta s' = \Delta s.$$

This finishes Example 1.8.1a.

Example 1.8.1b. Suppose that the temperature of the universe is

$$T = T_0 e^{-D} \tag{3}$$

where T_0 is a positive number and D is the distance of a point to the center of the universe. This form of the temperature is given in geometric terms, thus independent of coordinate systems. Somehow, it is convenient in many applications to have a coordinate system. So let us take one, and let the center of the universe C and an arbitrary point M to have coordinates

$$(c_1, c_2, c_3), \qquad (x_1, x_2, x_3),$$

hence a temperature given in this coordinate system as

$$T(x_1, x_2, x_3) = T_0 e^{-\sqrt{(x_1-c_1)^2+(x_2-c_2)^2+(x_3-c_3)^2}}. \tag{4}$$

Switching to a new coordinate system K', you will get new numbers for representing the same C and M, say:

$$(c_1', c_2', c_3'), \qquad (x_1', x_2', x_3'),$$

and the temperature representation is

$$T(x_1', x_2', x_3') = T_0 e^{-\sqrt{(x_1'-c_1')^2+(x_2'-c_2')^2+(x_3'-c_3')^2}}. \tag{5}$$

Here you should see I should not use the same T in (5) as in (4), because it is a different function than (4) since the $\mathbf{c}'$ is different from $\mathbf{c}$. Thus, as a mapping rule, the function in (5) is different and we call it a new name, T', for convenience.

But as we know, the distance between C and M is preserved when switching coordinate systems, thus the two functions T and T' yield the same value at M. Hence temperature is a scalar. This finishes Example 1.8.1b.

Examples of non-scalars are abound: components of higher order tensors are.

1.9. First-Order (Cartesian) Tensors (Vectors)

A first-order tensor is given by three components, and satisfies a certain transformation law.

Think of point B as displaced from point A. Then Δx_i are the *displacement.* We have seen that the displacement satisfies the law (2).

Definition. A vector (a.k.a first-order tensor) $\mathbf{A}$ is a quantity uniquely specified in any coordinate system by three real numbers (called the components of the vector) which transform under changes of the coordinate system according to the law

$$A_i' = \alpha_{i'k} A_k \tag{6}$$

where A_k, A_i' are the components of the vector in the old and new coordinate systems K and K' respectively, and $\alpha_{i'k}$ is the cosine of the angle between the i-th axis of K' and the k-th axis of K.

We remark that it is obvious that the zero vector $(0,0,0)$ is represented the same way in any coordinate system. Furthermore, this definition of vector is equivalent to the definition of a vector as a directed line segment. Lastly, we can use formula (6) to calculate the components of the representation of a vector in K' from the components of the representation in K.

Example 1.9a. A moving particle P has position coordinate $x_i(t)$ in a coordinate system K. The displacement

$$x_i(t+\Delta t) - x_i(t)$$

satisfies the law

$$x_i'(t+\Delta t) - x_i'(t) = \alpha_{i'k}(x_i(t+\Delta t) - x_i(t)) \tag{7}$$

by (2). Thus it determines a vector. We divide (7) by Δt (a scalar) to find

$$\frac{x_i'(t+\Delta t) - x_i'(t)}{\Delta t} = \alpha_{i'k}\frac{(x_i(t+\Delta t) - x_i(t))}{\Delta t}.$$

Taking the limit $\Delta t \to 0$ and using the definition of velocity

$$v_i(t) = \lim_{\Delta t \to 0} \frac{x_i(t+\Delta t) - x_i(t)}{\Delta t},$$

we find that

$$v_i'(t) = \alpha_{i'k} v_k(t).$$

So the velocity is a vector. Similarly the acceleration

$$a_k(t) = \frac{dv_k(t)}{dt}$$

is a vector. Multiplying the acceleration with the scalar mass m, and by Newton's second law, we find that the force

$$\mathbf{F} = m\mathbf{A}$$

is a vector.

1.10. Second-Order Tensors

Definition. A second-order tensor is a quantity uniquely specified by 9 real numbers (called the components of the tensor) which transform under changes of the coordinate system according to the law

$$A'_{ik} = \alpha_{i'l}\alpha_{k'm}A_{lm} \tag{8}$$

where A_{lm}, A'_{ik} are the components of the tensor in the old and new coordinate systems K and K' respectively, and $\alpha_{i'k}$ is the cosine of the angle between the i-th axis of K' and the k-th axis of K.

Remarks. **1.** We can use the transformation law to determine the coordinates of **A** from one system to another.

2. The zero tensor has zero coordinates in any coordinate system.

3. The components of a second-order tensor are often written as a matrix:

$$\mathbf{A} = (A_{ik}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

It can be regarded as a representation of a tensor with respect to a coordinate system.

4. Tensors of order higher than 2 cannot be represented by matrices.

Example 1.10a. Given two vectors **A** and **B**. There are nine products of a component of **A** with a component of **B**:

$$A_iB_k \qquad (i, k = 1, 2, 3).$$

Suppose we transform to a new coordinate system K', in which **A** and **B** have components A'_i and B'_k. Then

$$A'_i = \alpha_{i'l}A_l, \qquad B'_k = \alpha_{k'm}B_m$$

and hence

$$A'_iB'_k = \alpha_{i'l}\alpha_{k'm}A_lB_m.$$

This shows that A_iB_k is a second-order tensor. It is often denoted as $A \otimes B$.

More examples are in the text book.

——-End of Lecture 8———-