1.7. Coordinate transformations. (Continued)

Comment In physical sciences, a physical law is believed to hold with or without a coordinate system. For example, Newton's second law $\mathbf{F} = m\mathbf{a}$ holds no matter where you set up the origin of the coordinate system or how the coordinate axes are directed. It used to be the case that geometry was done without a coordinate system. Since DesCartes introduced the concept of a coordinate system, the study of geometry revolutionized and the study of the structure of the universe became possible. However, we need to learn a few things regarding the coordinate system to be able to utilize its power.

Coordinate transformations is the most important of such things. In geometry, the distance between two points M and P can be determined. The distance should be the same when measured using a des Cartes coordinate system. When one has two coordinate systems, the distance measured in the two systems should be equal. We can prove these by using the coordinate transformations. The coordinate transformations will also enable us to calculate the new coordinates of a physical quantity (e.g., a velocity vector) for given old coordinates.

Properties of $\alpha_{l'k}$.

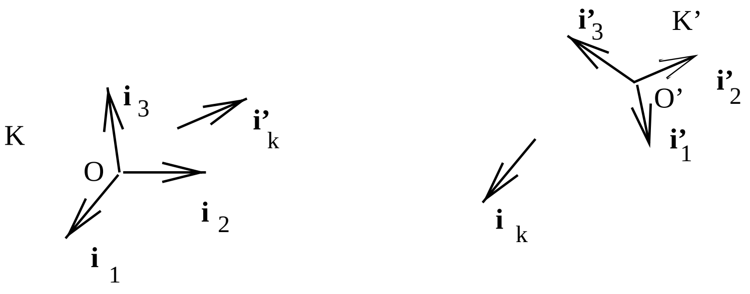
We note that the Kronecker delta in system K' is

$$\delta'_{kl} = \mathbf{i}'_k \cdot \mathbf{i}'_l = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases} \tag{1}$$

Note the expansions:

$$\mathbf{i}'_k = \alpha_{k'l}\mathbf{i}_l, \qquad \text{and} \qquad \mathbf{i}_k = \alpha_{l'k}\mathbf{i}'_l,$$

see Figure 1.7.2. And note the similarity between these expansions and the coordinate transformations in the previous paragraph.



(a) (b)

Figure 1.7.2. Calculations of the coeffieients.

The $(\alpha_{l'k})$ are often given as a 3×3 matrix

$$(\alpha_{l'k}) = \begin{pmatrix} \alpha_{1'1} & \alpha_{1'2} & \alpha_{1'3} \\ \alpha_{2'1} & \alpha_{2'2} & \alpha_{2'3} \\ \alpha_{3'1} & \alpha_{3'2} & \alpha_{3'3} \end{pmatrix}. \tag{2}$$

Note also

$$\begin{aligned} \delta_{km} &= \mathbf{i}_k \cdot \mathbf{i}_m &&= \alpha_{l'k}\mathbf{i}'_l \cdot \alpha_{j'm}\mathbf{i}'_j &&= \alpha_{l'k}\alpha_{l'm} \\ \delta'_{km} &= \mathbf{i}'_k \cdot \mathbf{i}'_m &&= \alpha_{k'l}\mathbf{i}_l \cdot \alpha_{m'j}\mathbf{i}_j &&= \alpha_{k'l}\alpha_{m'l}. \end{aligned}$$

Thus we have the property

$$\begin{aligned} \alpha_{l'k}\alpha_{l'm} &= \delta_{km}, \\ \alpha_{k'l}\alpha_{m'l} &= \delta'_{km}. \end{aligned}$$

These properties simply say that the columns of the matrix $(\alpha_{l'k})$ are orthonormal, so are the rows of the matrix. Thus the matrix $(\alpha_{l'k})$ is an orthogonal matrix.

====End of Lecture 7======