1.6. Vector Fields (continued)

1.6.2. The divergence of a vector field.

Recall

$$\text{div } \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_1} + \frac{\partial A_3}{\partial x_3} = (\text{ new notation })\nabla \cdot \mathbf{A}.$$

Recall Gauss' Theorem:

$$\iiint_V \text{div } \mathbf{A}\, dV = \iint_{\partial V} \mathbf{A} \cdot \mathbf{n}\, dS.$$

Dividing the equation by the volume V and taking the limit $V \to 0$ so that V shrinks to a point $\mathbf{x}$, we find

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = \lim_{V \to 0} \frac{1}{V} \iint_{\partial V} \mathbf{n} \cdot \mathbf{A}\, dS.$$

So we have found a coordinate-independent representation of the divergence. Using the definition of flux, we see that the integral over ∂V is the total flux through the surface ∂V. This total flux over volume V is the flux per unit volume. In the limit $V \to 0$, the limiting value measures the flux production per unit volume at the point $\mathbf{x}$. This is the real meaning of the divergence. If it is positive, then it is called a *source*. If it is negative, then it is called a *sink*. If it is zero in a domain, then there is no source or sink, and it is called divergence free.

Example 1.6.2a. Let

$$\mathbf{A}(\mathbf{r}) = q\frac{\mathbf{r}}{r^3}$$

where r denotes the norm of $\mathbf{r}$. It is an exercise that

$$\text{div } \mathbf{A} = 0 \qquad (r \neq 0)$$

at every point except $r = 0$. We are interested to find the flux through a sphere S centered at the origin. We know that the unit exterior normal to a sphere is given by

$$\mathbf{n} = \frac{\mathbf{r}}{r}.$$

So we have

$$\begin{aligned} \iint_S \mathbf{A} \cdot \mathbf{n}\, dS &= \iint_S q\tfrac{\mathbf{r}}{r^3} \cdot \tfrac{\mathbf{r}}{r}\, dS \\ &= q \iint_S \tfrac{1}{r^2} dS \\ &= 4\pi q. \end{aligned} \tag{1}$$

If $q > 0$, it is a source (fountain). If $q < 0$, it is a sink. By Gauss' Theorem, we can see that the flux through any surface is $4\pi q$ if the surface encloses the origin. The flux is zero if the surface does not enclose the origin. We also see that the flux is the same no matter how small the surface is as long as it contains the origin. This vector field is smooth every where away from the origin, and the origin is a point source/sink. See Figure 1.6.2.

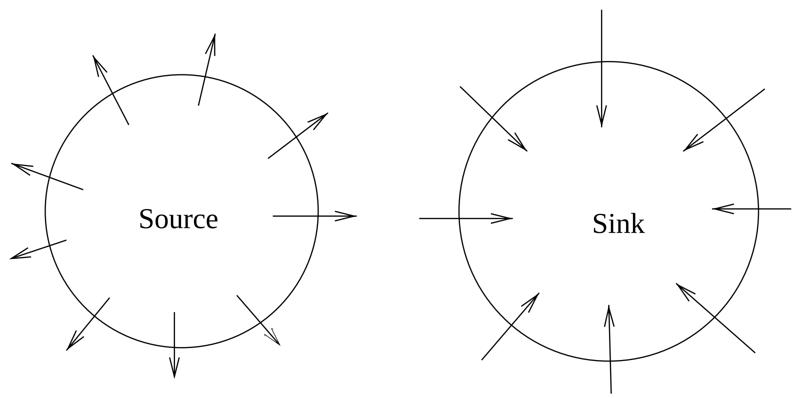

Figure 1.6.2. Flux and source/sink.

1.6.3. The curl of a vector field.

Recall we have introduced the curl of a vector in association with Stokes' Theorem:

$$\begin{aligned} \text{curl } \mathbf{A} &= \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}, \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}, \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \\ &= \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}. \end{aligned}$$

We state without proof that the curl has a coordinate-independent representation:

$$\text{curl } \mathbf{A} = \lim_{V \to 0} \frac{1}{V} \iint_{\partial V} \mathbf{n} \times \mathbf{A}\, dS \quad (= \nabla \times \mathbf{A}).$$

"You can see a lot by looking – Jogi Berra". Can you see a common theme in the three formulas for $\nabla\phi, \nabla \cdot \mathbf{A}, \nabla \times \mathbf{A}$? Once you treat $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ as a vector, many formulas involving first derivatives are a lot easier to memorize. For example, once you know the determinant formula for $\mathbf{A} \times \mathbf{B}$, you can replace $\mathbf{A}$ by ∇ to find $\nabla \times \mathbf{B}$.

We see the real meaning of the curl in the next example.

Example 1.6.3a. Consider a rigid body rotating about a fixed point O with angular velocity $\mathbf{w}$. See Figure 1.6.3. The velocity of a point with position vector $\mathbf{r}$ is given by

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}.$$

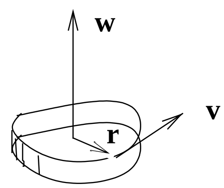

Figure 1.6.3. Curl is twice angular velocity.

Let us calculate the curl of $\mathbf{v}$.

$$\begin{aligned} \mathrm{curl}_1 \mathbf{v} &= \partial_{x_2} v_3 - \partial_{x_3} v_2 \\ &= \partial_{x_2}(w_1 x_2 - w_2 x_1) - \partial_{x_3}(w_3 x_1 - w_1 x_3) \\ &= 2w_1 \end{aligned} \tag{2}$$

(w is independent of $\mathbf{r}$), and similarly (homework)

$$\begin{aligned} \mathrm{curl}_2 \mathbf{v} &= 2w_2 \\ \mathrm{curl}_3 \mathbf{v} &= 2w_3. \end{aligned} \tag{3}$$

It follows that

$$\mathrm{curl}\ \mathbf{v} = 2w.$$

That is, the curl of the velocity field of a rotating body equals twice the angular velocity of the body.

1.7. Coordinate transformations.

We deal with coordinate transformations between rectangular coordinate systems, which play an important role in the definition of tensors.

Preliminary remark on tensors. Tensors are physical quantities that exist independent of coordinate systems. Scalar quantities are called zeroth-order tensors (e.g., temperature); vectors are called first-order tensors (e.g., velocity); second-order tensors can all be represented by 3×3 matrices (e.g., the stress tensor), but not all 3×3 matrices are tensors. True telephone numbers versus a string of 11 digits 1(812)855-8724 provides a metaphor. (That is, not every such a string of digits is a telephone number.) Tensors of orders greater than 2 cannot be represented by 3×3 matrices. An n-th-order tensor requires 3^n real numbers and is invariant under change of coordinate systems. The requirement of the invariance is natural since physical observables are invariant under change of coordinate systems.

Suppose we have two orthonormal bases:

$$\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \qquad \text{and} \qquad \mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3,$$

and two origin O and O' to form two rectangular coordinate systems K and K'. Let a point M in space have the representation

$$\begin{aligned} \mathbf{r} &= x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 \\ \mathbf{r}' &= x'_1\mathbf{i}'_1 + x'_2\mathbf{i}'_2 + x'_3\mathbf{i}'_3. \end{aligned} \tag{4}$$

Note the equations:

$$\begin{aligned} \mathbf{r} &= \mathbf{r}' + \mathbf{r}'_0, \qquad \mathbf{r}'_0 = \vec{OO'} \\ \mathbf{r}' &= \mathbf{r} + \mathbf{r}_0, \qquad \mathbf{r}_0 = \vec{O'O}, \end{aligned} \tag{5}$$

where the vector $\mathbf{r}'_0$ is a vector from O to O' and $\mathbf{r}_0 = -\mathbf{r}'_0$. See Figure 1.7.1.

(Figure 1.7.1.)

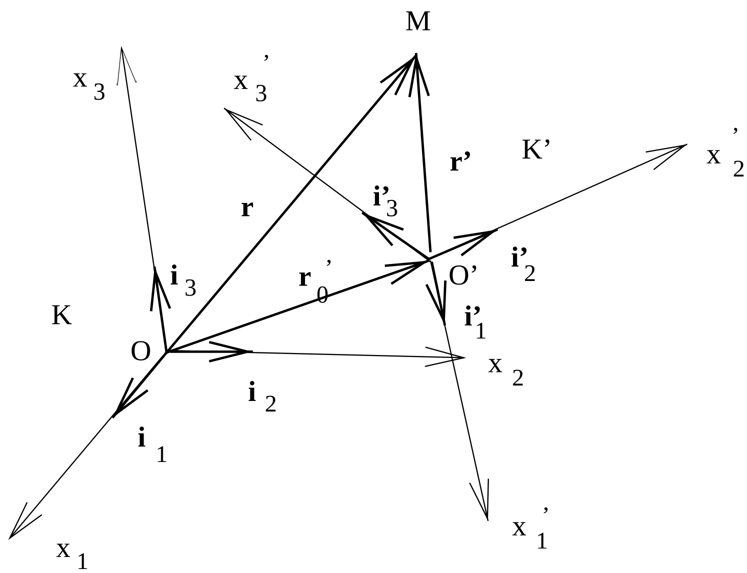

Figure 1.7.1. Coordinate transformation.

Now we use the summation convention: a repeated index in a product is automatically summed:

$$x_k \mathbf{i}_k = \sum_{k=1}^{3} x_k \mathbf{i}_k.$$

Thus, the equations in (5) can be written

$$\begin{aligned} x_k \mathbf{i}_k &= x'_k \mathbf{i}'_k + x'_{0k} \mathbf{i}_k \\ x'_k \mathbf{i}'_k &= x_k \mathbf{i}_k + x_{0k} \mathbf{i}'_k \end{aligned} \tag{6}$$

where x'_{0k} are the coordinates of $\mathbf{r}'_0$ with respect to the old system K, etc. Take inner product with $\mathbf{i}_l$ or $\mathbf{i}'_l$ in equations (6) and note the Kronecker delta function

$$\mathbf{i}_k \cdot \mathbf{i}_l = \delta_{kl} = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases} \tag{7}$$

We find

$$\begin{aligned} x_l &= x'_k (\mathbf{i}'_k \cdot \mathbf{i}_l) + x'_{0l} \\ x'_l &= x_k (\mathbf{i}_k \cdot \mathbf{i}'_l) + x_{0l}. \end{aligned} \tag{8}$$

We introduce new notations

$$\mathbf{i}'_k \cdot \mathbf{i}_l = 1 \cdot 1 \cdot \cos(\mathbf{i}'_k, \mathbf{i}_l) = \alpha_{k'l}. \tag{9}$$

Thus

$$\mathbf{i}_k \cdot \mathbf{i}'_l = \mathbf{i}'_l \cdot \mathbf{i}_k = \alpha_{l'k}. \tag{10}$$

Therefore

$$\begin{aligned} x_l &= \alpha_{k'l} x'_k + x'_{0l} \\ x'_l &= \alpha_{l'k} x_k + x_{0l}. \end{aligned} \tag{11}$$

The first equation of (11) is the transformation from K' to K, while the second equation of (11) is the inverse transformation from K' to K. Note the index summed in the second equation is the second index of α, while the index summed in the first equation of (11) is the first index.

If you want to know the parallel notation in matrix form, then the equation $x'_l = \alpha_{l'k} x_k + x_{0l}$ can be written as

$$x' = (\alpha_{l'k})\, x + x_0$$

where all x', x, x_0 are written in column vector form, and $(\alpha_{l'k})$ is written in matrix form.

——-End of Lecture 6——