

1.5. Scalar fields

Examples of scalar fields are the pressure function $p(\mathbf{r})$ and the temperature function $T(\mathbf{r})$ in a domain D .

A real function of \mathbf{r} in a domain is called a *scalar field*.

1.5.1. Gradient.

Let $h(x_1, x_2)$ denote the height of a mountain for (x_1, x_2) in a planar domain D . The set

$$\{(x_1, x_2) : h(x_1, x_2) = \text{a height } c\}$$

is called a *level curve*. Strictly speaking, level curves are curves on the ground. But sometimes, a level curve may mean a curve of a constant height on the surface of a mountain.

The **gradient** of $h(x)$ is defined as

$$\nabla h(x_1, x_2) = (\partial_{x_1} h, \partial_{x_2} h).$$

It is a vector. Its direction gives the direction for fastest change of h . It is normal to the level curve. See Figure 1.5.1.

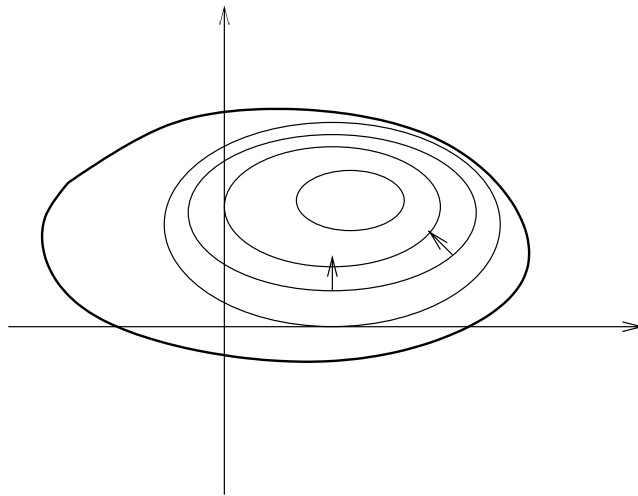


Figure 1.5.1. Level curves and the gradient.

In three dimensions, the gradient of a function $f(x_1, x_2, x_3)$ is defined similarly:

$$\nabla f(x_1, x_2, x_3) = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f).$$

It is perpendicular to the level surfaces:

$$\{(x_1, x_2, x_3) : f(x_1, x_2, x_3) = c\}$$

where c stands for a real number. The direction of ∇f points to the direction of fastest change of f . See Figure 1.5.2.

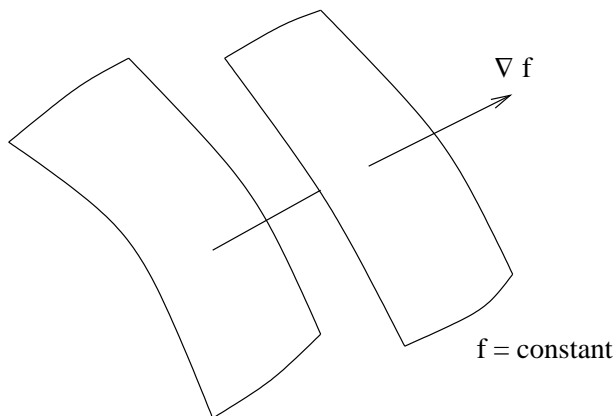


Figure 1.5.2. Level surfaces and the gradient.

Since we would like “to learn the REAL meaning of mathematical terms and how/when/why to use them”, quoted from the feedback of a former student, let us consider an example.

Example 1.5.1a. A spaceship is launched to a site near a cold, irregularly-shaped blackhole in the universe. Upon opening the door of the spaceship, the astronaut finds that the outside temperature is at 30 which is too cold for him and his ship. He quickly closes the door and finds that his location is at $(x_1, x_2, x_3) = (1, 1, 1)$ and the temperature distribution is

$$\phi(x_1, x_2, x_3) = 4x_1^2 + x_2^2 + 25x_3^2$$

(You can quickly verify that indeed $\phi(1, 1, 1) = 30$.) Help the astronaut find the best direction in which he should direct his ship to get to the warmer region.

Solution. The set $\{(x_1, x_2, x_3) : \phi = \text{constant}\}$ is a level surface. Our surface $\phi = 30$ is one of the level surfaces. To move quickly away from temperature 30, one

definitely needs to move away from the level set of 30. In fact he needs to move in the direction which is perpendicular to the level set. Fortunately, the gradient

$$\nabla\phi = (8x_1, 2x_2, 50x_3)$$

is a normal to the level surfaces. A normal to our surface at the point $(1, 1, 1)$ is

$$\nabla\phi(1, 1, 1) = (8, 2, 50).$$

See Figure 1.5.3. This direction is such that the temperature increases most rapidly. The opposite direction is such that the temperature decreases most rapidly.

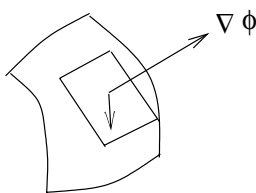


Figure 1.5.3. Gradient.

Summary: Gradient of a scalar field is normal to the level surfaces and the scalar increases most rapidly in the direction of the gradient among all possible directions. Proof will be given shortly.

1.5.2. Directional derivative.

We sometimes need to find the rate of change of a scalar in a scalar field in a given direction. For example, a NASA scientist wants to know the rate of change of air density along a chosen path for the re-entry of a space ship.

Given a point with position vector \mathbf{r} in a scalar field ϕ . Given also a unit vector \mathbf{l} . The rate of change of ϕ along \mathbf{l} at the point \mathbf{r} is defined as

$$\frac{d\phi}{d\mathbf{l}} = \lim_{\alpha \rightarrow 0} \frac{\phi(\mathbf{r} + \alpha\mathbf{l}) - \phi(\mathbf{r})}{\alpha}.$$

See Figure 1.5.4.

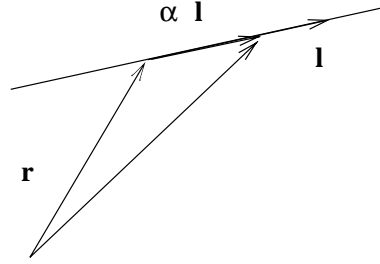


Figure 1.5.4. Directional derivative.

It can be shown that there holds

$$\frac{d\phi}{d\mathbf{l}} = \nabla\phi \cdot \mathbf{l}. \quad (1)$$

Mathematical proof of properties of gradient. Let $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{r} = (x_1, x_2, x_3)$. Then

$$\begin{aligned} \phi(\mathbf{r} + \alpha\mathbf{l}) - \phi(\mathbf{r}) &= \phi(x_1 + \alpha l_1, x_2 + \alpha l_2, x_3 + \alpha l_3) - \phi(x_1, x_2 + \alpha l_2, x_3 + \alpha l_3) \\ &\quad + \phi(x_1, x_2 + \alpha l_2, x_3 + \alpha l_3) - \phi(x_1, x_2, x_3 + \alpha l_3) \\ &\quad + \phi(x_1, x_2, x_3 + \alpha l_3) - \phi(x_1, x_2, x_3). \end{aligned} \quad (2)$$

We see that

$$\begin{aligned} &\frac{\phi(x_1 + \alpha l_1, x_2 + \alpha l_2, x_3 + \alpha l_3) - \phi(x_1, x_2 + \alpha l_2, x_3 + \alpha l_3)}{\alpha} \\ &= \frac{\phi(x_1 + \alpha l_1, x_2 + \alpha l_2, x_3 + \alpha l_3) - \phi(x_1, x_2 + \alpha l_2, x_3 + \alpha l_3)}{\alpha l_1} l_1 \end{aligned} \quad (3)$$

converges to

$$\frac{\partial\phi(x_1, x_2 + \alpha l_2, x_3 + \alpha l_3)}{\partial x_1} l_1 \Big|_{\alpha=0} = \frac{\partial\phi(x_1, x_2, x_3)}{\partial x_1} l_1 \quad (4)$$

as $\alpha \rightarrow 0$. Similarly, we can find the limits for the other two terms of difference in (2). In summary, we find that

$$\frac{d\phi}{d\mathbf{l}} = \frac{\partial\phi}{\partial x_1} l_1 + \frac{\partial\phi}{\partial x_2} l_2 + \frac{\partial\phi}{\partial x_3} l_3$$

which is (1).

Now we can derive the two properties of the gradient from (1). We see that the term $\nabla\phi \cdot \mathbf{l}$ achieves maximum when the angle between $\nabla\phi$ and \mathbf{l} is zero, i.e., $\mathbf{l} = \nabla\phi/|\nabla\phi|$, among all possible directions. So this is why the gradient gives the direction for most rapid increase. Furthermore, the least change is zero change, achieved at directions that are perpendicular to $\nabla\phi$, i.e., the angle between $\nabla\phi$ and \mathbf{l} is 90 degrees. We know intuitively that there is no change in a level surface. Thus the gradient is a normal to the level surfaces.

1.5.3. Coordinate-independent representation of gradient.

The representation is

$$\nabla\phi(\mathbf{x}) = \lim_{V \rightarrow 0} \frac{1}{V} \iint_{\partial V} \mathbf{n}\phi(\mathbf{y}) dS_{\mathbf{y}}$$

where V is a domain that contains the point \mathbf{x} and \mathbf{n} is the unit exterior normal to ∂V . We also use V to denote the volume of the domain V , a misnomer.

Mathematical proof of the coordinate-independent representation

Consider

$$\mathbf{A} = \mathbf{C}\phi$$

where \mathbf{C} is a constant vector. Let us apply Gauss' Theorem:

$$\iiint_V \operatorname{div} \mathbf{A} dV = \iint_{\partial V} \mathbf{A} \cdot \mathbf{n} dS.$$

Notice that $\operatorname{div} \mathbf{A} = \mathbf{C} \cdot \nabla\phi$. We find

$$\mathbf{C} \cdot \iiint_V \nabla\phi dV = \mathbf{C} \cdot \iint_{\partial V} \mathbf{n}\phi dS.$$

Since \mathbf{C} is arbitrary, we conclude that

$$\iiint_V \nabla\phi dV = \iint_{\partial V} \mathbf{n}\phi dS.$$

Dividing the equation by the volume V and taking the limit $V \rightarrow 0$, we have

$$\nabla\phi(\mathbf{x}) = \lim_{V \rightarrow 0} \frac{1}{V} \iint_{\partial V} \mathbf{n}\phi(\mathbf{y}) dS_{\mathbf{y}}.$$

1.6. Vector fields

1.6.1. Flux of a vector field.

Let S be a two-sided piecewise-smooth surface in a vector field $\mathbf{A}(\mathbf{r})$. Let \mathbf{n} be a unit normal to S . The flux of \mathbf{A} through an element dS is

$$\mathbf{A} \cdot \mathbf{n} dS.$$

The flux through S is

$$\iint_S \mathbf{A} \cdot \mathbf{n} dS.$$

See Figure 1.4.2 for an earlier definition of flux.

To figure out the real meaning of a flux, let us consider an example. Imagine a highway, an observational gate, and you are watching the passing cars. If no car is

moving (a complete stall), you see no car passing through your gate, we say that the flux is zero. Now suppose that all cars are moving at the same speed of 60 miles per hour. Then in one minute the first car you saw at the beginning of the minute has traveled one mile. If the density of cars on the highway is 1 car per mile, then you have seen 1 car in one minute passing through the gate. If the density is 2 cars per mile, then you have seen 2 cars passing. So both velocity and density plays roles. Now suppose that there are multiple lanes on the highway and the density is number of cars per mile per lane, then the number of cars passed depends on the number of gates you watch. That is, the cross length of the observational line plays a role. See Figure 1.6.1.

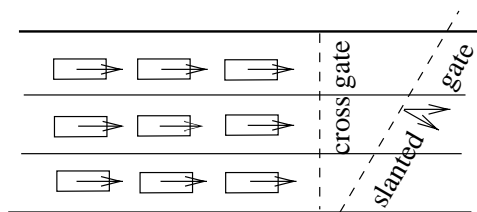


Figure 1.6.1. Flux associated with cars.

However, if the observational line is not perpendicular to the line of moving, then the real length of the observational line does not play a role; the projection of the observational line onto the line perpendicular to the lines of motion plays a role. This projection is the same as the projection of the velocity vector field onto the normal \mathbf{n} of the surface S (observational line). So in three dimensions, let \mathbf{v} be the velocity of fluid particles, let S be a surface, let ρ be the density of the fluid particles, then the quantity

$$\iint_S \rho \mathbf{v} \cdot \mathbf{n} dS$$

is the total mass of particles that have passed through S in unit time. Without the density factor, it is called the flux.

—End of Lecture 5—

(You can work on all the problems in Homework No. 2 by now. The remaining parts of Section 1.6, to be covered in the beginning of next lecture, will be helpful for Problems 8–10, but not essential.)