1.4. Theorems of Gauss, Green, and Stokes.

Recall the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Its magic is to reduce the domain of integration by one dimension. We want higher dimensional versions of this theorem.

We want two theorems like

$$\begin{aligned} \textstyle\iint_S (\text{integrand})\,dS &= \textstyle\oint_{\partial S}(\text{another integrand})\,d\ell \\ \textstyle\iiint_V (\text{integrand})\,dV &= \textstyle\iint_{\partial V}(\text{another integrand})\,dS. \end{aligned} \tag{1}$$

When S is a flat surface, the formula is called Green's Theorem. When S is curved, it is called Stokes' Theorem. The volume integral is called Gauss' Theorem.

Gauss' Theorem. *Let $P(x_1,x_2,x_3), Q(x_1,x_2,x_3), R(x_1,x_2,x_3)$ and all their partial derivatives be continuous in a given domain V with boundary ∂V. Then*

$$\iiint_V \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3}\right) dV = \iint_{\partial V} (P\cos(\mathbf{n},x_1) + Q\cos(\mathbf{n},x_2) + R\cos(\mathbf{n},x_3))\,dS.$$

Here $\mathbf{n}$ is the unit exterior normal to the surface ∂V. The term $(\mathbf{n},x_1)$ represents the angle between $\mathbf{n}$ and the x_1-axis, etc.

Note that the domain V can have holes: V can be a shell (a ball with another concentric, smaller ball removed, in which case the boundary of V consists of two disjoint parts: an exterior surface with normal pointing outside and an interior surface with exterior unit normal pointing to the origin). The boundary ∂V can be allowed to be piecewise smooth. But the functions $P(x_1,x_2,x_3), Q(x_1,x_2,x_3), R(x_1,x_2,x_3)$ and their derivatives are required to be continuous.

The more common form of Gauss' Theorem is in vector form. Let

$$\mathbf{A} = (P, Q, R).$$

Let the *divergence* of the vector $\mathbf{A}$ be

$$\text{div } \mathbf{A} = \frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3}.$$

Recall

$$\mathbf{n} = (n_1, n_2, n_3) = (\cos(\mathbf{n},x_1), \cos(\mathbf{n},x_2), \cos(\mathbf{n},x_3)).$$

Then Gauss' Theorem can be written in vector form:

$$\iiint_V \text{div } \mathbf{A}\, dV = \iint_{\partial V} \mathbf{A} \cdot \mathbf{n}\, dS.$$

The proof of Gauss' Theorem is omitted. We note that the divergence of a vector determines the source or sink of the vector field.

Green's Theorem. Given a planar region S bounded by a closed contour L. Suppose that $P(x_1, x_2)$ and $Q(x_1, x_2)$ and all their partial derivatives are continuous in the union $S \cup L$. Then

$$\iint_S (\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2})\, dS = \oint_{\partial S} P\, dx_1 + Q\, dx_2,$$

where L is traversed in the direction such that S appears to the left of an observer moving along L.

Stokes Theorem Given a surface S bounded by a closed contour L. Suppose that all P, Q, R and their derivatives are continuous on the union of $S \cup L$. Then

$$\begin{aligned}&\textstyle\iint_S [(\frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3}) \cos(\mathbf{n}, x_1) + (\frac{\partial P}{\partial x_3} - \frac{\partial R}{\partial x_1}) \cos(\mathbf{n}, x_2) + (\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}) \cos(\mathbf{n}, x_3)]\, dS \\ &\quad = \textstyle\oint_L P\, dx_1 + Q\, dx_2 + R\, dx_3,\end{aligned} \tag{2}$$

where $\mathbf{n}$ is the unit normal to S. Here L is traversed in the direction such that S appears to the left of an observer moving along L with the vector $\mathbf{n}$ at points near L pointing from the observer's feet to his/her head.

(Figure 1.4.1. Orientations in Stokes Theorem)

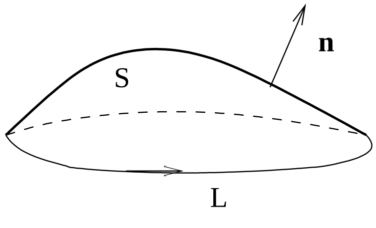

Figure 1.4.1. Orientations in Stokes' Theorem.

Stokes' Theorem in vector form. If we let

$$\mathbf{A} = (P, Q, R) = P\mathbf{i}_1 + Q\mathbf{i}_2 + R\mathbf{i}_3$$

and define

$$\begin{aligned}\text{curl } \mathbf{A} &= (\tfrac{\partial R}{\partial x_2} - \tfrac{\partial Q}{\partial x_3}, \tfrac{\partial P}{\partial x_3} - \tfrac{\partial R}{\partial x_1}, \tfrac{\partial Q}{\partial x_1} - \tfrac{\partial P}{\partial x_2}) \\ &= \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ P & Q & R \end{vmatrix},\end{aligned}$$

then Stokes' Theorem can be written as

$$\int\!\!\int_S \text{curl } \mathbf{A} \cdot \mathbf{n}\, dS = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}.$$

We see that the term $\oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}$ is the total circulation of the vector field $\mathbf{A}$ along ∂S. The term $\int\!\!\int_S \text{curl } \mathbf{A} \cdot \mathbf{n} dS$ is called the total *flux* of the vector field curl $\mathbf{A}$ through the surface S. In general the total *flux* of a vector $\mathbf{W}$ through a surface S is defined as

$$\int\!\!\int_S \mathbf{W} \cdot \mathbf{n}\, dS.$$

See Figure 1.4.2.

(Figure 1.4.2. Definition of flux)

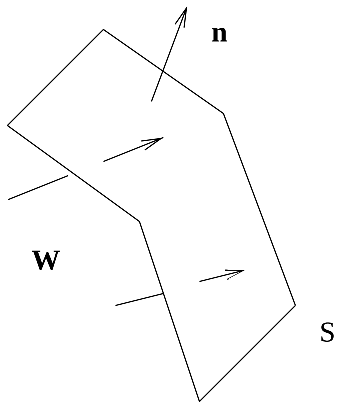

Figure 1.4.2. Flux of a vector field through a surface.

We will come back to the meaning of curl later. For now we note that the curl of a vector determines the rotation of the vector field. For a rigid body rotation, the curl of the velocity field is proportional to the angular velocity.

Supplemental reading

1.4.1. Simply connected domains.

We emphasize that Stokes' Theorem holds only when the vector field **A** and its curl are continuous on the union of the surface with its boundary. In general the continuity condition is verified in a domain D that contains S. Sometimes one may make mistakes in the relation of S with D. Let us consider the following question.

Let D be a domain. Suppose a vector **A** and all its derivatives are continuous in D. Suppose further that curl $\mathbf{A} = \mathbf{0}$ in D. Can we then use Stokes' Theorem to conclude that $\oint_L \mathbf{A} \cdot d\mathbf{r} = 0$ for any contour L within D?

The answer is yes if D is a solid ball or even a shell (a shell is the region between two concentric balls). But the answer is no if D is a torus. See Figure 1.4.3. To see why, we imagine a contour L that goes along the long circle of the torus. Now it is clear that we can not find a surface S whose boundary is L and lies entirely in the domain D. It may well be the case that the curl of **A** is not zero anymore outside of D. In this case, we do not have a surface S to apply Stokes's Theorem on.

(Figure 1.4.3. A torus.)

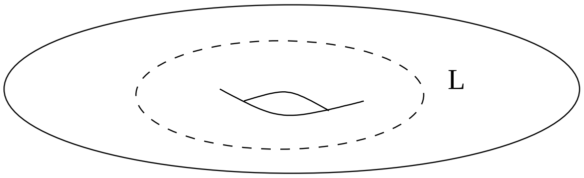

Figure 1.4.3. A torus and a contour that cannot shrink to a point within.

The difference between a torus and a shell or a ball can be characterized as follows. Any closed curve inside a ball can be shrunk within the ball to a point. But not every closed curve inside a torus can be shrunk within the torus to a point.

Definition. A domain is called **simply connected** if any closed curve inside the domain can be shrunk continuously to a point within the domain. A domain is called *multiply connected* if some closed curves cannot be shrunk within the domain to a point.

A torus is multiply connected. A ball is simply connected, so is a shell.

To see how a shell is simply connected, imagine a curve in a domain bounded by two parallel infinite planes. The curve can shrink within the plane to a point. Then imagine bending the plate so that a portion of the plate forms a portion of a shell.

See Figure 1.4.4.

(Figure 1.4.4. Definition of flux)

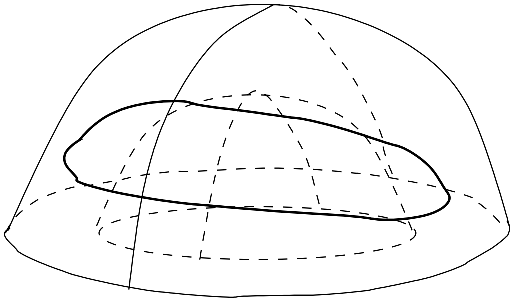

Figure 1.4.4. A shell is simply connected.

Stokes' Theorem applies to any contour L within a simply connected domain. In particular, circulation of a vector field along any closed curve within a simply connected domain is zero if the vector field and its curl are continuous and the curl vanishes at every point in the domain.

—– End of Lecture 4 ——