1.1.4. Product of three vectors.

Given three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. We list three products with formula

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C});$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B});$$

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where the entries are the coordinates of the three vectors. The first two products are called vector triple products, the third is called scalar triple product. The proof for the formulas for the vector triple products are complicated. But the proof for the formula for the scalar triple product is straightforward. The reader should be able to do it alone.

To remember the formulas for the two vector triple products, there is a quick way. You see that the final product of the first vector triple product will be perpendicular to $\mathbf{A} \times \mathbf{B}$, so it will lie in the plane spanned by $\mathbf{A}$ and $\mathbf{B}$. It is perpendicular to $\mathbf{C}$, so there will be no component in the $\mathbf{C}$ direction. So the first vector triple product is a linear combination of $\mathbf{A}$ and $\mathbf{B}$, not $\mathbf{C}$. The coefficients are the inner products of the remaining two vectors, with a minus sign for the second term; while the middle vector $\mathbf{B}$ is the first term.

Recall that the magnitude (length) of $\mathbf{A} \times \mathbf{B}$ is the area of the parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$, and the inner product with $\mathbf{C}$ is this magnitude times $|\mathbf{C}| \cos \phi$, which is exactly the height of the parallelepiped with a “slanted height” $|\mathbf{C}|$ and a bottom parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$. Thus the magnitude of the scalar triple product is the volume of the parallelepiped formed by the three vectors. See Figure 1.1.3.3: Volume of the parallelepiped formed by three vectors.

One can form other triple products, but they all can be reduced quickly to one of the three mentioned here. One may notice that the second vector triple product can be reduced to the first vector product easily. So essentially there is only one vector triple product and one scalar triple product.

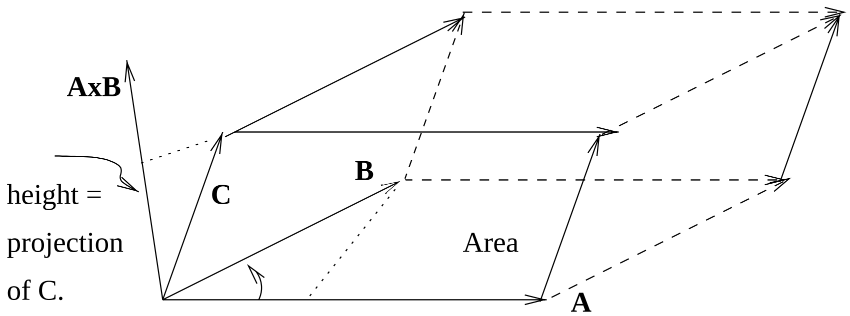

Figure 1.1.3.3. The volume of the parallelepiped is the magnitude of (AxB) \centerdot C.

1.2. Variable vectors.

1.2.1. Vector functions of a scalar argument.

Example 1.2.1a. $\mathbf{A}(t) = (\cos t, \sin t, 0), \quad (-\infty < t < \infty)$.

The graph (the collection of all the tips of the vector $\mathbf{A}(t)$) is a circle.

Example 1.2.1b. $\mathbf{A}(t) = (\cos t, \sin t, t), \quad (-\infty < t < \infty)$.

The graph is a helix.

Example 1.2.1c. $\mathbf{A}(t) = (3t, 2t, -t) = (3, 2, -1)t$. It represents a straight line.

The general formula for a straight line is

$$\mathbf{A}(t) = t\alpha + \beta$$

where α and β are numerical vectors independent of t.

1.2.2. The derivatives of a vector function.

Let

$$\mathbf{A}(t) = (A_1(t), A_2(t), A_3(t)) = A_1(t)\mathbf{i}_1 + A_2(t)\mathbf{i}_2 + A_3(t)\mathbf{i}_3.$$

Then

$$\frac{d\mathbf{A}(t)}{dt} = (\frac{dA_1(t)}{dt}, \frac{dA_2(t)}{dt}, \frac{dA_3(t)}{dt}) = \frac{dA_1(t)}{dt}\mathbf{i}_1 + \frac{dA_2(t)}{dt}\mathbf{i}_2 + \frac{dA_3(t)}{dt}\mathbf{i}_3.$$

Formal definition:

$$\mathbf{A}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(\mathbf{t})}{\Delta t}.$$

(Figure 1.2.1. Derivative: pay attention to the direction of the difference quotient.)

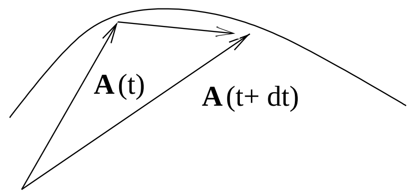

Figure 1.2.1. Derivative: tangent direction.

Example 1.2.2a. Let $\mathbf{r}(t)$ be the position vector of a moving particle. Then

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t) \qquad \text{is the velocity;}$$

$$\frac{d^2\mathbf{r}(t)}{dt^2} = \frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t) \qquad \text{is the acceleration.}$$

Example 1.2.2b. Consider $\mathbf{r}(t) = (\cos t, \sin t)$. Then $\mathbf{r}'(t) = (-\sin t, \cos t)$. Note that the derivative is tangent to the graph of $\mathbf{r}(t)$. Consider $\mathbf{R}(t) = 2(\cos t, \sin t)$. Then $\mathbf{R}'(t) = 2(-\sin t, \cos t)$. Notice that the magnitude of the derivative is twice as large. See Figure 1.2.2.

(Figure 1.2.2. Derivative: direction and magnitude.)

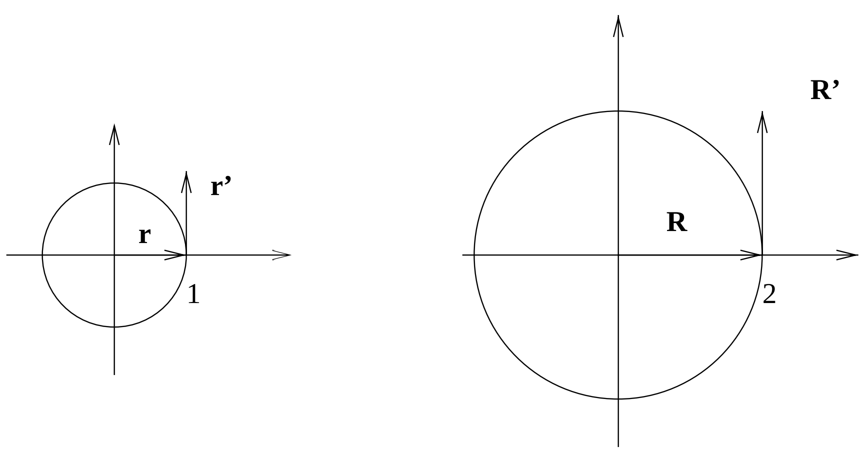

Figure 1.2.2. Derivative: direction and magnitudes.

Simple rules:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{A} \pm \mathbf{B}) &= \frac{d}{dt}\mathbf{A} \pm \frac{d}{dt}\mathbf{B}, \\
\frac{d}{dt}(c\mathbf{A}) &= \frac{dc}{dt}\mathbf{A} + c\frac{d}{dt}\mathbf{A}, \\
\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &= \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt}. \\
\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) &= \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}.
\end{aligned} \tag{1}$$

1.2.3. The integral of a vector function.

The integral of a vector function is defined also component-wise.

Let

$$\mathbf{A}(t) = (A_1(t), A_2(t), A_3(t)) = A_1(t)\mathbf{i}_1 + A_2(t)\mathbf{i}_2 + A_3(t)\mathbf{i}_3.$$

Then

$$\mathbf{B}(t) = \int \mathbf{A}(t)\,dt = (\int A_1(t)\,dt, \int A_2(t)\,dt, \int A_3(t)\,dt).$$

So far we have covered in the lectures the following sections of our text book (Vector and Tensor Analysis with Applications, by Borisenko etc.) 1.2.3, 1.4, 1.5, and 1.7.

1.3. Vector fields

We have mentioned the magnetic field, which is defined as a domain in which a vector of magnetism is defined at every point. Another example is the velocity field in a stream: each water droplet has a velocity. See Figure 1.3.1: Velocity field.

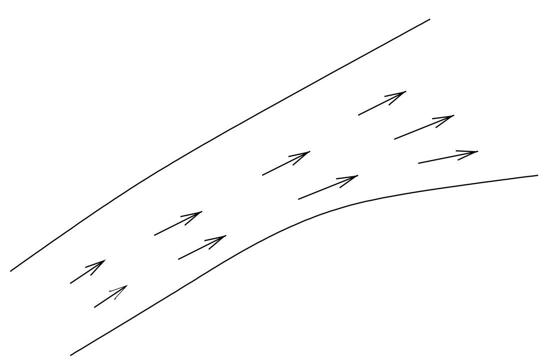

Figure 1.3.1. The velocity field of a stream.

Generally, a vector field is a domain Ω and a vector function $\mathbf{A}(\mathbf{r})$ defined in it. Furthermore, a vector field may be time-dependent: $\mathbf{A}(\mathbf{r}, t)$.

1.3.1. Line integrals and circulation.

We introduce an integral that gives work done by a force field or circulation of velocity around a loop.

Let $\mathbf{A}(\mathbf{r})$ be a vector field with domain Ω. Let M_1M_2 be a curve in the domain directed from M_1 to M_2. Chop the curve into many small pieces, say n pieces. One typical piece is denoted by the end points $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$. See Figure 1.3.2. The work done in this piece is approximately $\mathbf{A}(\mathbf{r}_i)\cdot\Delta\mathbf{r}_i$, where $\Delta\mathbf{r}_i=\mathbf{r}_{i+1}-\mathbf{r}_i$, if we imagine that the vector field is a force field. This can also be interpreted as the flow of the vector field in the direction of $\Delta\mathbf{r}_i$. We sum over all such pieces and take the limit as all $\Delta\mathbf{r}_i\to 0$ to define the *line integral*:

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{A}(\mathbf{r}_i)\cdot\Delta\mathbf{r}_i &= \int_{M_1M_2}\mathbf{A}(\mathbf{r})\cdot d\mathbf{r}\\ &= \int_{M_1M_2}A_1dx_1+A_2dx_2+A_3dx_3.\end{aligned}\tag{2}$$

Here the notation is $\mathbf{A}=A_1\mathbf{i}_1+A_2\mathbf{i}_2+A_3\mathbf{i}_3$.

Line integrals give either total work done by the vector field, or total flow of the vector field along the curve M_1M_2 in the direction specified.

Total *circulation* around a contour L is defined as

$$\Gamma=\oint_L\mathbf{A}\cdot d\mathbf{r}.$$

(Figure 1.3.2. Definition of the line integral.)

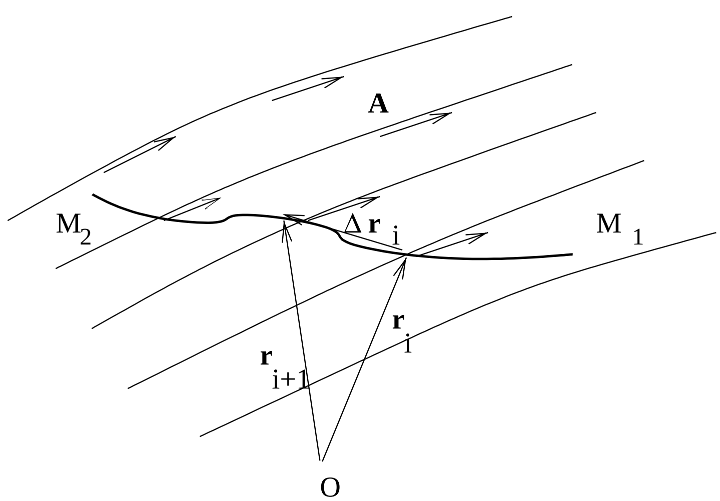

Figure 1.3.2. Definition of the line integral.

Example 1.3.1a. Let $\mathbf{A} = (-x_2, x_1, 0)$. Let L be the unit circle: $x_1^2 + x_2^2 = 1, x_3 = 0$ and counter-clockwise. Then

$$\begin{aligned} \Gamma &= \oint_L \mathbf{A} \cdot d\mathbf{r} &&= \oint_L -x_2 dx_1 + x_1 dx_2 \\ & && \quad (L : x_1 = \cos\theta, x_2 = \sin\theta, 0 \le \theta \le 2\pi) \\ & &&= \int_0^{2\pi} -\sin\theta d\cos\theta + \cos\theta d\sin\theta \\ & &&= \int_0^{2\pi} \sin^2\theta d\theta + \cos^2\theta d\theta \\ & &&= \int_0^{2\pi} d\theta = 2\pi. \end{aligned} \tag{3}$$

See Figure 1.3.3.

(Figure 1.3.3. Examples of circulation and line integrals.)

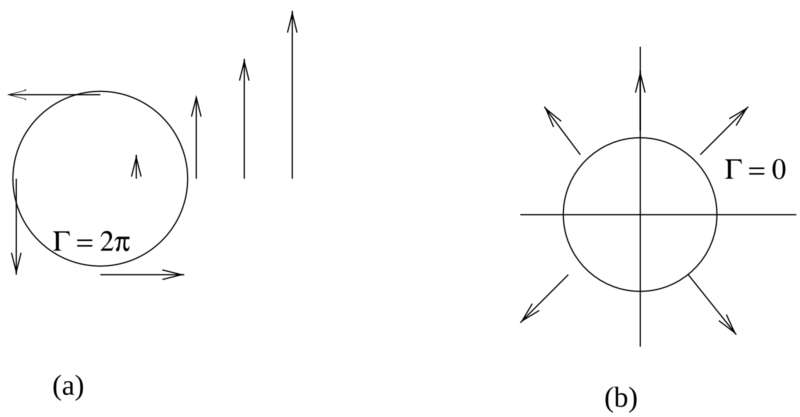

Figure 1.3.3. Examples of circulation and line integrals.

Example 1.3.1b. Let $\mathbf{A} = (x_1, x_2, 0)$ and L as before. Then

$$\begin{aligned} \Gamma &= \oint_L \mathbf{A} \cdot d\mathbf{r} &&= \oint_L x_1 dx_1 + x_2 dx_2 \\ & && \quad (L : x_1 = \cos\theta, x_2 = \sin\theta, 0 \le \theta \le 2\pi) \\ & &&= \int_0^{2\pi} \cos\theta d\cos\theta + \sin\theta d\sin\theta \\ & &&= \int_0^{2\pi} -\cos\theta\sin\theta d\theta + \sin\theta\cos\theta d\theta \\ & &&= \int_0^{2\pi} 0 d\theta = 0. \end{aligned} \tag{4}$$

Example 1.3.1c. Let $\mathbf{A} = (x_1, x_2, 0)$ and L be the line segment: $0 \le x_1 \le 1, x_2 = 0$ directed toward the x_1-axis. Then

$$\Gamma = \oint_L \mathbf{A} \cdot d\mathbf{r} = \int_0^1 x_1 dx_1 = \frac{1}{2} x_1^2|_0^1 = \frac{1}{2}. \tag{5}$$

–End of Lecture 3 —