1.16. Grad, div, and curl in orthogonal curvilinear coordinate systems.

In this section we derive the expressions of various vector concepts in an orthogonal curvilinear coordinate system.

Let (u_1, u_2, u_3) be such a system:

$$u_i = \phi_i(x_1, x_2, x_3). \qquad (i = 1, 2, 3).$$

Let

$$x_i = f_i(u_1, u_2, u_3)$$

be the inverse transformation. We introduce the normalized coordinate tangent vectors:

$$\mathbf{u}_i = \frac{1}{h_i}\frac{\partial \mathbf{R}}{\partial u_i} \quad (\text{ no summation }) \; i = 1, 2, 3,$$

where $h_i = |\partial \mathbf{R}/\partial u_i|$. Assume that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is right-handed so that the Jacobian is positive.

1.16.1. Gradient of a scalar field.

Let $F(x_1, x_2, x_3)$ be a scalar field in a rectangular system. We know that ∇F is a vector, which can be represented as a linear combination of any basis. So let

$$\nabla F = F_1\mathbf{u}_1 + F_2\mathbf{u}_2 + F_3\mathbf{u}_3.$$

We need to find (F_1, F_2, F_3). We recall from Section 1.5.3(of Lecture 5) the coordinate-independent formula

$$\nabla F(P_0) = \lim_{V\to 0}\frac{1}{V}\iint_{\partial V} \mathbf{n}F(\mathbf{y})\, dS_{\mathbf{y}} \tag{1}$$

where V is a domain that contains the point P_0 and $\mathbf{n}$ is the unit exterior normal to ∂V. By the way, we have also the formulas

$$\nabla \cdot \mathbf{F}(P_0) = \lim_{V\to 0}\frac{1}{V}\iint_{\partial V} \mathbf{n}\cdot\mathbf{F}(\mathbf{y})\, dS_{\mathbf{y}}$$

for the divergence $(\nabla\cdot)$ of a vector field $\mathbf{F}$ and and

$$\nabla \times \mathbf{F}(P_0) = \lim_{V\to 0}\frac{1}{V}\iint_{\partial V} \mathbf{n}\times\mathbf{F}(\mathbf{y})\, dS_{\mathbf{y}}$$

for the curl $(\nabla\times)$ which we will use for the representations of div and curl. The three formulas certainly have striking uniformity. Back to our gradient representation, we take V to be an elementary "curvilinear parallelepiped" of volume

$$ds_1\, ds_2\, ds_3 = h_1h_2h_3\, du_1\, du_2\, du_3$$

with faces perpendicular to the coordinate curves, see Figure 1.16.1.

During the lecture I messed up a crucial step. Here is what is needed. The four essential corners of the curvilinear parallelepiped are given approximately by P_0, $P_0 + \mathbf{u}_1\, du_1$, $P_0 + \mathbf{u}_2\, du_2$, and $P_0 + \mathbf{u}_3\, du_3$. Although the displacement from P_0 to $P_0 + \mathbf{u}_1\, du_1$ is $\mathbf{u}_1\, du_1$, the actual (geometric) distance is $ds_1 = h_1\, du_1$.

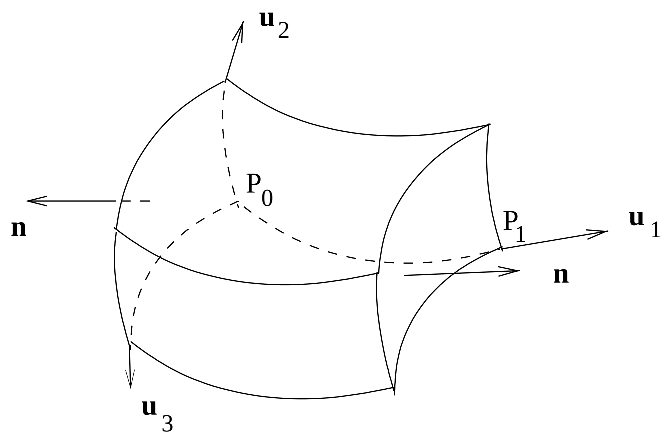

Figure 1.16.1. Curvilinear parallelepiped.

To calculate the surface integral (1), we first note that there are six sides. For the side that passes through P_0 and perpendicular to the u_1-axis, we have the approximate value

$$-\mathbf{u}_1\, F(P_0)\, h_2 du_2\, h_3 du_3,$$

where the surface area element is $ds_2\, ds_3 = h_2 du_2\, h_3 du_3$. The integral on the surface that is parallel to the previous surface is approximately

$$\mathbf{u}_1\, F(P_0 + du_1 \mathbf{u}_1)\, h_2 du_2\, h_3 du_3,$$

where $P_1 = P_0 + du_1 \mathbf{u}_1$ is the position of P_0 with an increment du_1 along the u_1-coordinate axis. Combining these two sides and note that the volume of the element is

$$V = ds_1\, ds_2\, ds_3 = h_1 h_2 h_3\, du_1 du_2 du_3,$$

the average becomes

$$\frac{(-F(P_0) + F(P_0 + du_1 \mathbf{u}_1)) h_2 h_3\, du_2 du_3}{h_1 h_2 h_3\, du_1 du_2 du_3} \longrightarrow \frac{1}{h_1} \frac{\partial F(P_0)}{\partial u_1} \mathbf{u}_1$$

as $V \to 0$. Similarly we can calculate the other four sides. In summary, we find

$$\nabla F = \frac{1}{h_1}\frac{\partial F}{\partial u_1}\mathbf{u}_1 + \frac{1}{h_2}\frac{\partial F}{\partial u_2}\mathbf{u}_2 + \frac{1}{h_3}\frac{\partial F}{\partial u_3}\mathbf{u}_3.$$

Theorem The del operator has the formula

$$\nabla = \mathbf{u}_1\frac{1}{h_1}\frac{\partial}{\partial u_1} + \mathbf{u}_2\frac{1}{h_2}\frac{\partial}{\partial u_2} + \mathbf{u}_3\frac{1}{h_3}\frac{\partial}{\partial u_3}.$$

Example 1.16a Find the expression of ∇ in cylindrical coordinates.

Solution. Let $u_1 = r, u_2 = \theta, u_3 = z$. It is right-handed. We have $h_1 = 1, h_2 = r, h_3 = 1$. Also

$$\mathbf{u}_1 = \cos\theta\,\mathbf{i}_1 + \sin\theta\,\mathbf{i}_2,\ \mathbf{u}_2 = -\sin\theta\,\mathbf{i}_1 + \cos\theta\,\mathbf{i}_2,\ \mathbf{u}_3 = \mathbf{i}_3.$$

Thus

$$\nabla = \mathbf{u}_1\frac{\partial}{\partial r} + \mathbf{u}_2\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{u}_3\frac{\partial}{\partial z}.$$

Example 1.16b. Find the gradient of $f = xyz$ in the cylindrical coordinates.

Solution. We have $f = r^2 z\sin\theta\cos\theta$. Thus

$$\nabla f = \mathbf{u}_1 2rz\sin\theta\cos\theta + \mathbf{u}_2 rz(\cos^2\theta - \sin^2\theta) + \mathbf{u}_3 r^2\sin\theta\cos\theta.$$

1.16.2. Divergence. We let

$$\mathbf{F} = F_1\mathbf{u}_1 + F_2\mathbf{u}_2 + F_3\mathbf{u}_3.$$

Then we can find, similar to the previous section, that

$$\text{div }\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_1h_3) + \frac{\partial}{\partial u_3}(F_3h_1h_2)\right].$$

Example 1.16c Derive the formula for the Laplacian Δ defined as $\Delta = \text{div }\nabla$.

Solution. Consider an $\mathbf{F} = \nabla f$. We have

$$\begin{aligned}\Delta f &= \text{div }\nabla f \\ &= \tfrac{1}{h_1h_2h_3}\left[\tfrac{\partial}{\partial u_1}(\tfrac{h_2h_3}{h_1}\tfrac{\partial f}{\partial u_1}) + \tfrac{\partial}{\partial u_2}(\tfrac{h_1h_3}{h_2}\tfrac{\partial f}{\partial u_2}) + \tfrac{\partial}{\partial u_3}(\tfrac{h_1h_2}{h_3}\tfrac{\partial f}{\partial u_3})\right].\end{aligned} \tag{2}$$

1.16.3. The curl.

Similarly, for

$$\mathbf{F} = F_1\mathbf{u}_1 + F_2\mathbf{u}_2 + F_3\mathbf{u}_3,$$

we can derive the formula

$$\text{curl } \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{u}_1 & h_2 \mathbf{u}_2 & h_3 \mathbf{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}.$$

Appendix: Useful expressions

I. In cylindrical coordinates

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = z$$
$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

there hold

$$\begin{aligned} \text{grad } f &= \tfrac{\partial f}{\partial r}\mathbf{u}_r + \tfrac{1}{r}\tfrac{\partial f}{\partial \theta}\mathbf{u}_\theta + \tfrac{\partial f}{\partial z}\mathbf{u}_z, \\ \text{div } \mathbf{A} &= \tfrac{1}{r}\tfrac{\partial}{\partial r}(rA_r) + \tfrac{1}{r}\tfrac{\partial A_\theta}{\partial \theta} + \tfrac{\partial A_z}{\partial z}, \\ \text{curl } \mathbf{A} &= \left(\tfrac{1}{r}\tfrac{\partial A_z}{\partial \theta} - \tfrac{\partial A_\theta}{\partial z}\right)\mathbf{u}_r + \left(\tfrac{\partial A_r}{\partial z} - \tfrac{\partial A_z}{\partial r}\right)\mathbf{u}_\theta + \\ &\quad + \tfrac{1}{r}\left(\tfrac{\partial}{\partial r}(rA_\theta) - \tfrac{\partial A_r}{\partial \theta}\right)\mathbf{u}_z, \\ \Delta f &= \tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\tfrac{\partial f}{\partial r}\right) + \tfrac{1}{r^2}\tfrac{\partial^2 f}{\partial \theta^2} + \tfrac{\partial^2 f}{\partial z^2}, \end{aligned}$$

where

$$\mathbf{u}_r = \cos\theta\, \mathbf{i}_1 + \sin\theta\, \mathbf{i}_2, \ \mathbf{u}_\theta = -\sin\theta\, \mathbf{i}_1 + \cos\theta\, \mathbf{i}_2, \ \mathbf{u}_z = \mathbf{i}_3$$

is the local orthonormal basis, and $\mathbf{A}$ has components A_r, A_θ, A_z with respect to this basis.

II. In spherical coordinates. See text book by Borisenko, p174.

======End of Lecture 14 =========