1.15.2. Arclength and orthogonal curvilinear coordinate systems.

We assume that the three vectors

$$(\frac{\partial \mathbf{R}}{\partial u_1}, \frac{\partial \mathbf{R}}{\partial u_2}, \frac{\partial \mathbf{R}}{\partial u_3})$$

form a right-handed basis; i.e., the vector product $\frac{\partial \mathbf{R}}{\partial u_1} \times \frac{\partial \mathbf{R}}{\partial u_2}$ has positive inner product with $\frac{\partial \mathbf{R}}{\partial u_3}$. In this case, the Jacobian $\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}$ is positive.

We derive arclength formula in curvilinear coordinate systems. Consider the position vector

$$\mathbf{R} = x_i \mathbf{i}_i = f_i(u_1, u_2, u_3)\mathbf{i}_i.$$

We have

$$\begin{aligned} (ds)^2 &= d\mathbf{R} \cdot d\mathbf{R} \\ &= (\textstyle\sum_{i=1}^3 \frac{\partial \mathbf{R}}{\partial u_i}\, du_i) \cdot (\textstyle\sum_{j=1}^3 \frac{\partial \mathbf{R}}{\partial u_j}\, du_j) \\ &= (\frac{\partial \mathbf{R}}{\partial u_i} \cdot \frac{\partial \mathbf{R}}{\partial u_j}) du_i du_j \\ &= g_{ij}\, du_i\, du_j \end{aligned} \tag{1}$$

where we have introduced

$$g_{ij} = \frac{\partial \mathbf{R}}{\partial u_i} \cdot \frac{\partial \mathbf{R}}{\partial u_j}$$

which is called the *metric tensor*.

From here one can pursue the study of general metric tensors, which are used for example in general relativity. For us, we choose to be more specific. We say that the curvilinear coordinate system is *orthogonal curvilinear* if the triple vectors $\partial \mathbf{R}/\partial u_i (i = 1, 2, 3)$ are mutually orthogonal. For orthogonal curvilinear coordinate systems, the directions and magnitudes of $\partial \mathbf{R}/\partial u_i (i = 1, 2, 3)$ can still vary. Let us define

$$h_i = |\frac{\partial \mathbf{R}}{\partial u_i}| \quad (i = 1, 2, 3).$$

Then we have

$$g_{ij} = \begin{cases} h_i^2, & i = j, \\ 0, & i \neq j. \end{cases}$$

Example 1.15b. The transformation relating the cylindrical coordinates (r, θ, z) to the rectangular cartesian coordinates (x, y, z) is defined by the equations

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arccos \frac{x}{\sqrt{x^2+y^2}} \quad (\text{ or } \arcsin \frac{y}{\sqrt{x^2+y^2}}) \\ z &= z. \end{aligned}$$

It is defined for all (x, y, z) except for the origin.

We find

$$\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & 0 \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{r}.$$

The inverse is

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

which is valid for all (r, θ, z).

Let us identify the coordinate surfaces and coordinate curves. Refer to Figure 1.15.2.

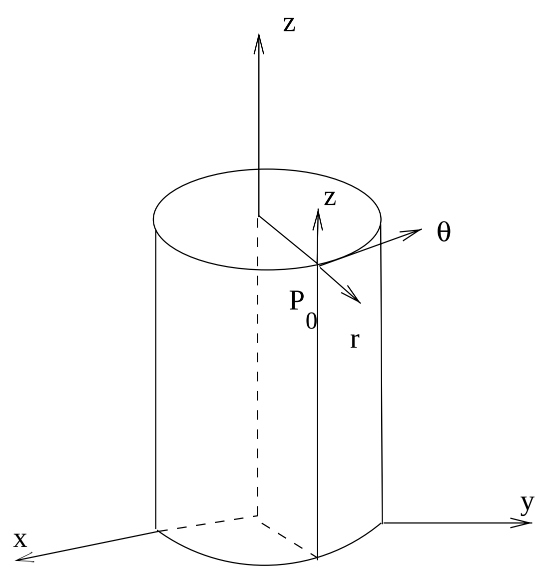

Figure 1.15.2. Cylindrical coordinate system.

Coordinate surfaces: The coordinate surface $r = r_0$ is the surface of a cylinder passing through a point P_0, and extending to infinity in both the positive and negative directions of z-axis. The coordinate surface $\theta = \theta_0$ is a half plane starting at the z-axis and extending to infinity. The coordinate surface $z = z_0$ is the plane passing through P_0 and perpendicular to the z-axis.

Coordinate curves: The r-coordinate curve is a ray starting on the z-axis, passing through the point P_0, and parallel to the xy-plane. The θ-coordinate curve is a circle passing through the point P_0, and parallel to the xy-plane. The z-coordinate curve is a straight line parallel to the old z-axis.

We have the position vector

$$\mathbf{R}(r,\theta,z) = r\cos\theta\,\mathbf{i}_1 + r\sin\theta\,\mathbf{i}_2 + z\,\mathbf{i}_3.$$

Tangent vectors to the coordinate curves are

$$\begin{aligned} \tfrac{\partial \mathbf{R}}{\partial r} &= \cos\theta\ \mathbf{i}_1 + \sin\theta\ \mathbf{i}_2 \\ \tfrac{\partial \mathbf{R}}{\partial \theta} &= -r\sin\theta\ \mathbf{i}_1 + r\cos\theta\ \mathbf{i}_2 \\ \tfrac{\partial \mathbf{R}}{\partial z} &= \mathbf{i}_3. \end{aligned}$$

Let $u_1 = r, u_2 = \theta, u_3 = z$, then the three tangent vectors form a right-handed orthogonal curvilinear coordinate system. We find that $g_{ij} = 0$ for $i \neq j$, and

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1.$$

Thus the distance formula is

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2.$$

Example 1.15c. The spherical coordinates $u_1 = r, u_2 = \phi, u_3 = \theta$ are defined by

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \phi &= \arccos \tfrac{z}{\sqrt{x^2+y^2+z^2}} \\ \theta &= \arccos \tfrac{x}{\sqrt{x^2+y^2}}. \end{aligned}$$

$(r \geq 0, 0 \leq \phi < \pi, 0 \leq \theta < 2\pi)$. Refer to Figure 1.15.3 for the variables.

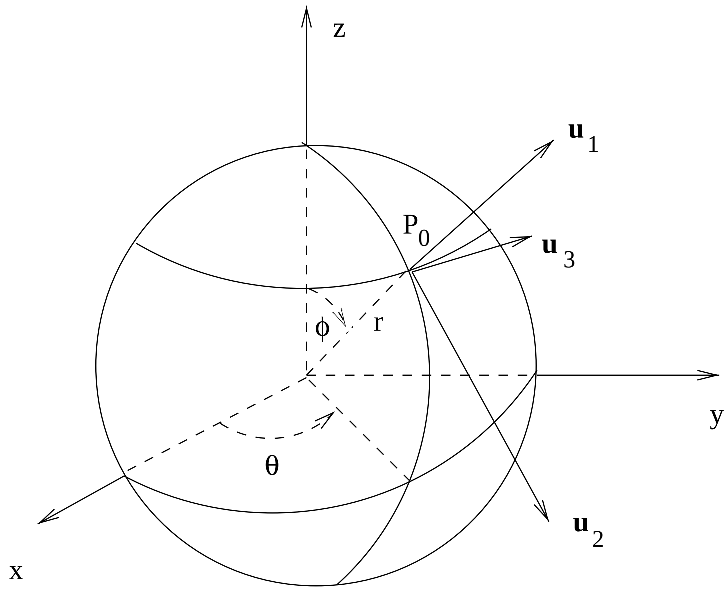

Figure 1.15.3. Spherical coordinate system.

We can calculate the Jacobian

$$\frac{\partial(r,\phi,\theta)}{\partial(x,y,z)} = \begin{vmatrix} \frac{r}{x} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & -\frac{x^2+y^2}{r^2} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} = \frac{1}{r^2\sin\phi}.$$

The inverse is

$$\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi. \end{aligned}$$

The Jacobian is

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = r^2\sin\phi.$$

The position vector is

$$\mathbf{R}(r,\phi,\theta) = r\sin\phi\cos\theta\ \mathbf{i}_1 + r\sin\phi\sin\theta\ \mathbf{i}_2 + r\cos\phi\ \mathbf{i}_3.$$

The three tangent vectors are

$$\begin{aligned}
\frac{\partial \mathbf{R}}{\partial r} &= \sin\phi\cos\theta\ \mathbf{i}_1 + \sin\phi\sin\theta\ \mathbf{i}_2 + \cos\phi\ \mathbf{i}_3 \\
\frac{\partial \mathbf{R}}{\partial \phi} &= r\cos\phi\cos\theta\ \mathbf{i}_1 + r\cos\phi\sin\theta\ \mathbf{i}_2 - r\sin\phi\ \mathbf{i}_3 \\
\frac{\partial \mathbf{R}}{\partial \theta} &= -r\sin\phi\sin\theta\ \mathbf{i}_1 + r\sin\phi\cos\theta\ \mathbf{i}_2.
\end{aligned}$$

They are mutually orthogonal. We have

$$\begin{aligned}
g_{11} &= h_1^2 &&= 1 \\
g_{22} &= h_2^2 &&= r^2 \\
g_{33} &= h_3^2 &&= r^2\sin^2\phi.
\end{aligned}$$

The distance formula is

$$(ds)^2 = (dr)^2 + (rd\phi)^2 + (r\sin\phi d\theta)^2.$$

For the volume element, we have

$$dV = dr \cdot rd\phi \cdot r\sin\phi\, d\theta = r^2\sin\phi\, dr\, d\phi\, d\theta.$$

In the next lecture we calculate the grad, div, and curl in orthogonal curvilinear coordinate systems.

====== End of Lecture 13 ========