

1.15. Curvilinear Coordinate Systems.

We need curvilinear coordinate systems in applications. The spherical coordinate system is an example of curvilinear coordinate systems.

Let u_1, u_2, u_3 denote new coordinates and suppose that they are related to the cartesian coordinates x_1, x_2, x_3 by the equations

$$u_i = \phi_i(x_1, x_2, x_3) \quad (i = 1, 2, 3). \quad (1)$$

Assume that ϕ_i have continuous first-order derivatives in a domain D of the x -space and there holds the condition

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_1}{\partial x_3} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \frac{\partial\phi_2}{\partial x_3} \\ \frac{\partial\phi_3}{\partial x_1} & \frac{\partial\phi_3}{\partial x_2} & \frac{\partial\phi_3}{\partial x_3} \end{vmatrix} \neq 0 \quad (2)$$

in the domain. This determinant is called the *Jacobian* of the transformation from \mathbf{x} to \mathbf{u} .

The nonvanishing condition (2) ensures that it is possible to determine (x_1, x_2, x_3) in terms of the coordinates (u_1, u_2, u_3) ; i.e., there exist functions $f_i(u_1, u_2, u_3)$ ($i = 1, 2, 3$) such that

$$x_i = f_i(u_1, u_2, u_3) \quad (3)$$

where f_i are defined in D determined from (1). Moreover, f_i have continuous first-order derivatives for which

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} \neq 0$$

in D . The functions (f_1, f_2, f_3) define the *inverse* transformation of (1). It is important to note that the Jacobians satisfy the relation

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} \cdot \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = 1. \quad (4)$$

For a proof of (4), see Homework # 5, problem 1.

See the Supplemental Materials at the end of this lecture for why the nonvanishing condition (2) ensures invertibility.

Now let P be any point in D with coordinate (x_1, x_2, x_3) and let the numbers u_1, u_2, u_3 be determined by (1). We call the ordered triple of numbers (u_1, u_2, u_3) the

curvilinear coordinates of the point P . The equations in (1) are called the *coordinate transformation*, and they are said to define a *curvilinear coordinate system* in D . It follows that the Jacobian of a coordinate transformation is the reciprocal of the Jacobian of its inverse.

Example 1.15a. Consider the transformation from the rectangular cartesian coordinates (x, y) on a plane to the polar coordinates (r, θ) defined by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}} \text{ (or } = \arcsin \frac{y}{\sqrt{x^2 + y^2}} \text{)}$$

where \arccos is chosen such that a unique θ in $0 \leq \theta < 2\pi$ exists so that $\cos \theta = x/\sqrt{x^2 + y^2}$ and $\sin \theta = y/\sqrt{x^2 + y^2}$. The domain D is all points except the origin. The Jacobian is

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix} = \frac{1}{r}.$$

In the above calculation we find the partial derivative $\partial r / \partial x$ as follows: From

$$r^2 = x^2 + y^2$$

we find

$$2rr_x = 2x.$$

Dividing by $2r$ we find $r_x = x/r$. We can use $\cos \theta = x/r$ to find the derivative θ_x , etc. Thus the inverse exists except at the origin, the inverse is

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Note that the inverse is defined for all (r, θ) . We can calculate

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

It is clear that the product of the two Jacobians is 1.

1.15.1. Coordinate surfaces, coordinate curves, and local basis.

Let $P_0 = (x_1^0, x_2^0, x_3^0)$ be a point in D with coordinates (u_1^0, u_2^0, u_3^0) in the curvilinear coordinate system. We call the surface

$$\phi_i(x_1, x_2, x_3) = u_i^0$$

the i -th coordinate surface passing through P_0 ($i = 1, 2, 3$). The intersection of two coordinate surfaces, say,

$$\phi_1(x_1, x_2, x_3) = u_1^0, \quad \phi_2(x_1, x_2, x_3) = u_2^0$$

is called the u_3 -coordinate curve. See Figure 1.15.1.

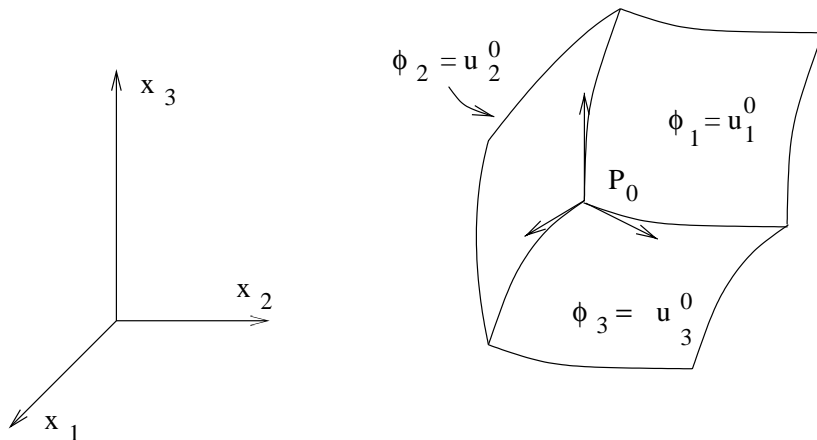


Figure 1.15.1. Coordinate surfaces, coordinate curves, and local basis.

We next derive a basis at the point P_0 . The position vector of an arbitrary point P is

$$\mathbf{R}(u_1, u_2, u_3) = x_i \mathbf{i}_i = f_i(u_1, u_2, u_3) \mathbf{i}_i. \quad (5)$$

If we set $u_2 = u_2^0, u_3 = u_3^0$, then the resulting vector function $\mathbf{R}(u_1, u_2^0, u_3^0)$ represents the u_1 -curve. On this curve u_1 is the parameter. It follows that the derivative

$$\frac{\partial \mathbf{R}}{\partial u_1}$$

represents the tangent vector to this curve. Likewise, we have

$$\frac{\partial \mathbf{R}}{\partial u_2}, \quad \frac{\partial \mathbf{R}}{\partial u_3}$$

representing the tangent vectors to the u_2 - and u_3 -curves respectively.

We next prove the identity

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} = \frac{\partial \mathbf{R}}{\partial u_1} \cdot \left(\frac{\partial \mathbf{R}}{\partial u_2} \times \frac{\partial \mathbf{R}}{\partial u_3} \right). \quad (6)$$

From (5), we have

$$\frac{\partial \mathbf{R}}{\partial u_i} = \frac{\partial x_1}{\partial u_i} \mathbf{i}_1 + \frac{\partial x_2}{\partial u_i} \mathbf{i}_2 + \frac{\partial x_3}{\partial u_i} \mathbf{i}_3. \quad (i = 1, 2, 3)$$

Now use the scalar triple product formula (Lecture 3, Sect 1.1.4), the determinant of a matrix is the same as the determinant of its transpose, and the definition of Jacobian, we find (6) is true.

Hence at each point where (6) is not zero, the three tangent vectors

$$\left(\frac{\partial \mathbf{R}}{\partial u_1}, \frac{\partial \mathbf{R}}{\partial u_2}, \frac{\partial \mathbf{R}}{\partial u_3} \right) \quad (7)$$

form a parallelepiped of non-zero volume, thus are *linearly independent* and form a basis.

Every vector or vector field at each point can then be represented in terms of this basis (7). Unlike the unit vectors $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$, however, this new basis varies from point to point in space. For this reason, we call (7) a *local basis*.

==End of Lecture 12=====

Supplemental Materials. If the derivative $f'(x)$ is not zero at a point x_0 , then the function $y = f(x)$ is invertible at $x = x_0$. The corresponding condition for multivariables is the Jacobian. Let me explain. First we know that in order to solve x from $Ax = b$ we need the square matrix A to be invertible: $\det(A) \neq 0$. Now we use Taylor expansion for each of u_i at a point P :

$$u_i(x_1, x_2, x_3) = \phi_i(P) + \frac{\partial \phi_i(P)}{\partial x_j} (x_j - P_j) + \text{high order terms}$$

where $i = 1, 2, 3$. So we see we can solve for $x_j - P_j$ provided that the Jacobian is not zero, since the high order terms are not important at point P .