1.11. High-Order Tensors.

By a tensor of order n is meant a quantity uniquely specified by 3^n real numbers (the components of the tensor) which transform under changes of coordinate systems according to the law

$$A'_{i_1 i_2 \cdots i_n} = \alpha_{i'_1 k_1} \alpha_{i'_2 k_2} \cdots \alpha_{i'_n k_n} A_{k_1 k_2 \cdots k_n}$$

where $A_{k_1 k_2 \cdots k_n}$, $A'_{i_1 i_2 \cdots i_n}$ are the components of the vector in the old and new coordinate systems K and K' respectively, and $\alpha_{i'k}$ is the cosine of the angle between the i-th axis of K' and the k-th axis of K.

Example 1.11a. If $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are three vectors, then the $3^3 = 27$ quantities

$$D_{ijk} = A_i B_j C_k$$

form a tensor of order 3. The proof is omitted, but see an exercise.

Example 1.11b. Suppose one second-order tensor A_{ik} is a linear function of another second-order tensor B_{ik}, such that

$$A_{ik} = \lambda_{iklm} B_{lm},$$

then λ_{iklm} form a fourth-order tensor. Proof is omitted.

1.12. Tensor Algebra.

1.12.1. Addition. We can add any two tensors of the same order, the sum is a tensor of the same order, whose components are the sums of the corresponding components of the two tensors. For example, tensor A_{ik} and tensor B_{ik} can be added to give a tensor C_{ik}:

$$C_{ik} = A_{ik} + B_{ik}.$$

1.12.2. Multiplication. We can multiply any number of tensors of arbitrary orders. The product of two tensors, for example, is a tensor whose order is the sum of the orders of the two tensors, and whose components are products of a component of one tensor with any component of the other tensor. The product of two second-order tensors A_{ik} with B_{lm}, for example, is a fourth-order tensor C_{iklm} with components

$$C_{iklm} = A_{ik} B_{lm}.$$

Our product of tensors is also called *outer product.*

1.12.3. Contraction of Tensors.

Summing a tensor of order n ($n \geq 2$) over two of its indices is called contraction. For example, summing over the first and second indices of a third-order tensor

$$A_{iik} = A_{11k} + A_{22k} + A_{33k}$$

gives a vector. This is called contraction in the first and second indices. Contraction in both indices of a second-order tensor B_{ij} gives a scalar

$$B_{ii} = B_{11} + B_{22} + B_{33}.$$

Another example is A_{iki} gives another vector.

Contraction can be done many times.

Inner product. Multiplying two or more tensors and then contracting the product with respect to indices belonging to different factors is often called an *inner product* of the given tensors. For example, $A_{ik}B_k$, A_iB_i, and $\lambda_{iklm}B_{lm}$ are all inner products. But $A_{ii}B_k$ is not an inner product.

1.13. Symmetry Properties of Tensors.

A tensor $S_{ikl\cdots}$ (of order 2 or higher) is said to be *symmetric* in the first and second indices (say) if

$$S_{ikl\cdots} = S_{kil\cdots}.$$

It is *antisymmetric* in the first and second indices (say) if

$$S_{ikl\cdots} = -S_{kil\cdots}.$$

Antisymmetric tensors are also called *skewsymmetric* or *alternating* tensors. The Kronecker δ_{ik} is a symmetric second-order tensor since

$$\delta_{ik} = \mathbf{i}_i \cdot \mathbf{i}_k = \mathbf{i}_k \cdot \mathbf{i}_i = \delta_{ki}.$$

The stress tensor p_{ik} is symmetric. But the tensor

$$C_{ik} = A_iB_k - A_kB_i$$

is antisymmetric. It can be shown easily that an antisymmetric second-order tensor has an matrix like this:

$$(C_{ik}) = \begin{pmatrix} 0 & C_{12} & C_{13} \\ -C_{12} & 0 & C_{23} \\ -C_{13} & -C_{23} & 0 \end{pmatrix}.$$

That is $C_{ik} = 0$ for $i = k$ for an antisymmetric tensor.

We note that any second-order tensor T_{ik} can be represented as a sum of a symmetric tensor and an antisymmetric tensor:

$$T_{ik} = S_{ik} + A_{ik}$$

where

$$\begin{aligned} S_{ik} &= \tfrac{1}{2}(T_{ik} + T_{ki}) \\ A_{ik} &= \tfrac{1}{2}(T_{ik} - T_{ki}). \end{aligned}$$

1.14. The alternating tensor of third order: Pseudotensors

Given a coordinate system K with the basis vectors $\mathbf{i}_i, (i = 1, 2, 3)$. Let us consider the quantities:

$$\epsilon_{jkl} = (\mathbf{i}_j \times \mathbf{i}_k) \cdot \mathbf{i}_l.$$

We have been assuming that our coordinate system K is always right-handed, i.e., the thumb of the right hand points to the direction of $\mathbf{i}_3$ if we position our right hand so that our four fingers can rotate from $\mathbf{i}_1$ to $\mathbf{i}_2$. In this case, we can calculate to find that

$$\epsilon_{jkl} = \begin{cases} 1, & \text{if } j,k,l \text{ is a cyclic permutation of } 1,2,3. \\ -1, & \text{if } j,k,l \text{ is a cyclic permutation of } 2,1,3. \\ 0, & \text{otherwise.} \end{cases}$$

More precisely, we have

$$\begin{aligned} \epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{213} &= \epsilon_{132} = \epsilon_{321} = -1, \end{aligned}$$

and all others with repeated indices $\epsilon_{111} = \epsilon_{112} = \cdots$ are zero. We verify for example that

$$\epsilon_{123} = (\mathbf{i}_1 \times \mathbf{i}_2) \cdot \mathbf{i}_3 = \mathbf{i}_3 \cdot \mathbf{i}_3 = 1,$$

and

$$\epsilon_{213} = (\mathbf{i}_2 \times \mathbf{i}_1) \cdot \mathbf{i}_3 = -\mathbf{i}_3 \cdot \mathbf{i}_3 = -1,$$

and

$$\epsilon_{113} = (\mathbf{i}_1 \times \mathbf{i}_1) \cdot \mathbf{i}_3 = \mathbf{0} \cdot \mathbf{i}_3 = 0.$$

Under orthogonal coordinate transformations from this K to another right-handed system K', we can show that ϵ_{jkl} transform like a third-order tensor. Proof is given below in smaller font. It is sometimes denoted as δ_{jkl}, a reminder that it is a generalization of the Kronecker δ_{jk}.

We note further that the permutation tensor ϵ_{jkl} is antisymmetric in any pair of indices:

$$\epsilon_{jkl} = -\epsilon_{kjl}; \quad \epsilon_{jkl} = -\epsilon_{jlk}; \quad \epsilon_{jkl} = -\epsilon_{lkj}.$$

With two swaps, we have

$$\epsilon_{jkl} = -\epsilon_{kjl} = \epsilon_{klj}, \qquad \text{etc.}$$

Because of this, it is often called *the alternating tensor of third order.*

We note an interesting application of the alternating tensor. The vector product $\mathbf{A} \times \mathbf{B}$ has a representation:

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k.$$

That is, the outer product of ϵ_{jkl} with vectors $\mathbf{A}$ and $\mathbf{B}$ yields a tensor of order 5; when contracted twice, the result is a tensor of order 1.

Pseudotensors. But occasionally we need to use left-handed coordinate systems. In this case the thumb of the left hand points to the direction of $\mathbf{i}_3$ if we position our left hand so that our four fingers can rotate from $\mathbf{i}_1$ to $\mathbf{i}_2$. In a left handed coordinate system the vector product of $\mathbf{A} \times \mathbf{B}$ is defined by the left hand rule; i.e., the direction of $\mathbf{A} \times \mathbf{B}$ has the direction so that the three vectors $\mathbf{A}, \mathbf{B}$, and $\mathbf{A} \times \mathbf{B}$ follow the left hand rule. For either handedness, *the rule of the direction of the product* $\mathbf{A} \times \mathbf{B}$ *is such that the three vectors* $\mathbf{A}, \mathbf{B}$, *and* $\mathbf{A} \times \mathbf{B}$ *have the same handedness as the coordinate system*. This way all the formula for vector product hold for both kinds of coordinate systems. In particular, the formula

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

is valid in both kinds of coordinate system.

A coordinate system transformation may change the handedness. We have allowed for these transformations in our definition of tensors of all orders.

However there are tensor-like quantities that change slightly differently from the laws of tensors. For example, let us calculate the changes in ϵ_{jkl}. Let K' be a coordinate system with the basis vectors $\mathbf{i}'_1 = \mathbf{i}_1, \mathbf{i}'_2 = \mathbf{i}_2, \mathbf{i}'_3 = -\mathbf{i}_3$ and the same origin. By definition we have

$$\epsilon'_{123} = (\mathbf{i}'_1 \times \mathbf{i}'_2) \cdot \mathbf{i}'_3.$$

Note that K' is now left handed. So the way to figure out the vector product $\mathbf{i}'_1 \times \mathbf{i}'_2$ is to use the left hand rule, so we find that

$$\mathbf{i}'_1 \times \mathbf{i}'_2 = \mathbf{i}'_3.$$

Thus

$$\epsilon'_{123} = 1.$$

Now let us calculate the term

$$\alpha_{1'l}\alpha_{2'm}\alpha_{3'n}\epsilon_{lmn}$$

which would be equal to ϵ'_{123} if ϵ_{jkl} were a third-order tensor. Note that the coordinate transformation coefficients are

$$(\alpha_{i'l}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus

$$\alpha_{1'l}\alpha_{2'm}\alpha_{3'n}\epsilon_{lmn} = \alpha_{1'1}\alpha_{2'2}\alpha_{3'3}\epsilon_{123} = -1.$$

We can do all the calculations to verify that there actually hold

$$\epsilon'_{ijk} = -\alpha_{i'l}\alpha_{j'm}\alpha_{k'n}\epsilon_{lmn}.$$

So ϵ_{jkl} is not a third-order tensor. This leads to the concept and definition of pseudotensors.

Definition of pseudotensors. A pseudotensor of order n has 3^n components $A_{k_1 k_2 \cdots k_n}$ that transform under changes of coordinate system according to the law

$$A'_{i_1 i_2 \cdots i_n} = \Delta \alpha_{i'_1 k_1} \alpha_{i'_2 k_2} \cdots \alpha_{i'_n k_n} A_{k_1 k_2 \cdots k_n}$$

where $A_{k_1 k_2 \cdots k_n}$, $A'_{i_1 i_2 \cdots i_n}$ are the components of the pseudovector in the old and new coordinate systems K and K' respectively, $\alpha_{i'k}$ is the cosine of the angle between the i-th axis of K' and the k-th axis of K, Δ is 1 if K and K' have the same handedness, and Δ is -1 if K and K' have different handedness.
Note that a change of coordinate system is called a *proper transformation* if it preserves the handedness. It is called an *improper transformation* if it reverses the handedness. Pseudotensors are also called *tensor densities*.

We can verify that the permutation tensor ϵ_{jkl} is a third-order pseudotensor. It is called *the unit pseudotensor of order three*. Since it appears in many physical and geometrical situations, it also has the name *Levi-Civita tensor density*. Its another name is *the alternating pseudotensor of third order*.
Lastly we note that the vector product $\mathbf{A} \times \mathbf{B}$, which does not transform as an ordinary vector, has a pseudotensor representation:

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k.$$

That is, pseudotensors can be multiplied to yield pseudotensors. Higher-order pseudotensors can be contracted to form pseudotensors. In the current situation, the outer product of ϵ_{jkl} with ordinary vectors $\mathbf{A}$ and $\mathbf{B}$ yields a pseudotensor of order 5. When contracted twice, the result is a pseudotensor of order 1.
An ordinary first-order tensor is called a *polar vector*. Polar vectors transform under both types of changes of coordinate systems without the factor Δ. A first-order pseudotensor is called an *axial vector*. It is called axial because it has something to do with the axis of rotation associated in the product $\mathbf{v} = \vec{\omega} \times \mathbf{r}$.