1.10.2. The moment of inertia tensor. (Self-reading) Consider a rigid body system of n particles with coordinates $(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}), j = 1, 2, \cdots, n$ and mass m_j in a coordinate system K with origin O. The quantities

$$I_{ik} = \sum_{j=1}^{n} m_j (\delta_{ik} x_l^{(j)} x_l^{(j)} - x_i^{(j)} x_k^{(j)})$$

are called the *moment of inertia tensor* (about the origin O). It is a second-order tensor. It is used in physics in

$$\omega_k I_{ik} = L_i$$

where

$$\mathbf{L} = \sum_{j=1}^{n} m_j (\mathbf{r}^{(j)} \times \mathbf{v}^{(j)})$$

is the angular momentum and $\vec{\omega}$ is the angular velocity:

$$\mathbf{v}^{(j)} = \vec{\omega} \times \mathbf{r}^{(j)}.$$

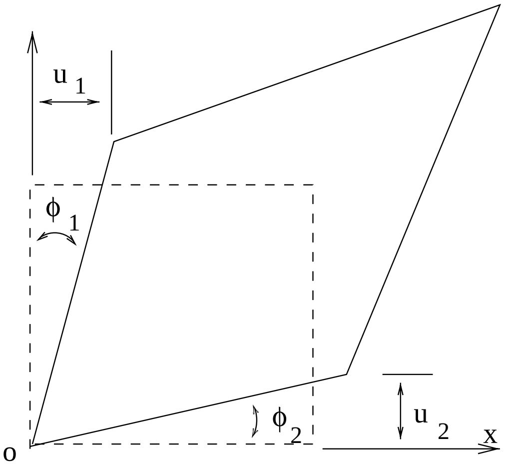

Figure 1.10.4. Shear displacements u_1 and u_2 and angles of shear phi_1 and phi_2 for a sheared volume element.

1.10.3. The Deformation (Strain) Tensor.

Let $\mathbf{u}(\mathbf{r})$ be the displacement of a point with position vector $\mathbf{r}$. Then the quantities

$$u_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

form a second-order tensor, called the deformation tensor. It is more often called the *strain tensor.* As an example, the shear strain, u_{12}, indicated in Figure 1.10.4, is related to the angles of shear by the relation

$$u_{12} = \frac{1}{2}(\phi_1 + \phi_2).$$

Another example is the normal strain u_{ii}, the stretch per unit length in the direction $\mathbf{i}_i$, shown in Figure 1.10.5.

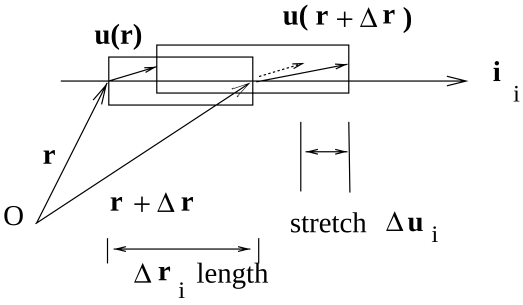

Figure 1.10.5. Normal Strain is stretch per unit length.

We show that the strain tensor is a second-order tensor. Recall the coordinate transformation for the displacements

$$u_i' = \alpha_{i'k} u_k.$$

The coordinate transform and its inverse are

$$x_i' = \alpha_{i'k} x_k + x_{0i}; \quad x_i = \alpha_{k'i} x_k' + x_{0i}'.$$

From these and the chain rule we find the differentiation formula

$$\frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'} \frac{\partial}{\partial x_j} = \alpha_{i'j} \frac{\partial}{\partial x_j}.$$

By definition, we have

$$u_{ij}' = \frac{1}{2}(\frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'}).$$

By the insertion of the expressions of u_i' and the differentiation formula, we find

$$u_{ij}' = \frac{1}{2}\alpha_{j'm} \frac{\partial}{\partial x_m}(\alpha_{i'l} u_l) + \frac{1}{2}\alpha_{i'l} \frac{\partial}{\partial x_l}(\alpha_{j'm} u_m)$$

which can be contracted to

$$u'_{ij} = \alpha_{i'l}\alpha_{j'm}u_{lm}.$$

That is the proof.

By Hooke's law for linear elasticity, there are c_{ijkl} such that

$$p_{ij} = c_{ijkl}u_{kl}.$$

The c_{ijkl} are the elastic constants tensor, a fourth-order tensor.

1.10.4. The rate of Deformation Tensor.

Let $\mathbf{v}(M)$ be the velocity at a point M of a moving fluid. Then the quantities

$$v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

form a second-order tensor, called the rate of deformation tensor.

=====End of Lecture 10======