Provide twice as many details as is given in the proof of Theorem 1.4.3 (the $C^{2,\alpha}$ estimate of the Newtonian potential on a flat boundary) of the lecture notes, or Theorem 3.4 of the reference book by Ya-Zhe Chen. In particular, explain why the mollifier has to be special, why the constant $f(x_0)$ has to be subtracted early on in the proof, and why $D_x v \in C^{1,\alpha}(\mathbb{R}^n)$. 