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Hyperbolic Conservation Laws

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ABSTRACT. Abstract of the meeting is here.

Mathematics Subject Classification (2000): AMS-CLASSIFICATION.

Introduction by the Organisers

This meeting was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds . . .

Workshop: Hyperbolic Conservation Laws

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Abstracts

Two-Dimensional Riemann Problems for Conservation Laws

Yuxi Zheng

Consider the two-dimensional compressible Euler system

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + pI) = 0, \\ (\rho E)_t + \nabla \cdot (\rho E \mathbf{u} + p\mathbf{u}) = 0, \end{cases} \tag{1}$$

where ρ is the density, $\mathbf{u} = (u, v)$ is the velocity, p is the pressure, and $E = p\rho^{-1}/(\gamma - 1) + (u^2 + v^2)/2$ where $\gamma > 1$ is the gas constant. The other variables we use are speed of sound c such that $c^2 = \gamma p/\rho$ and entropy $S = p\rho^{-\gamma}$. Cauchy problems for (1) are open. Riemann problems for (1) are a current research topic, as they are reducible to involve fewer independent variables.

Riemann problems are Cauchy problems with special initial data that are constant along each ray from the origin. The four-wave Riemann problems are special Riemann problems whose initial data yield single waves along the interfaces of the four quadrants. A list of all possible configurations is available in [12, 10, 7, 14]. Numerical solutions to these configurations have been done in [1, 3, 4, 5, 10]. In particular, paper [3], being the latest, concentrates on Configurations A and B, the simplest two of the many cases, and reveals new details of the solutions.

In Config. A, the initial data $\{p_i, \rho_i, u_i, v_i\}$ in the ith-quadrant ($i = 1, 2, 3, 4$) are such that a forward planar rarefaction wave R^+_{ij} is there to connect the neighboring states i and j for each interface $(i, j) \in \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. The entropy turns out to be constant $S_i = S_j$, while (u, v, c) are related by

$$\begin{aligned} u_i - u_j &= 2(c_i - c_j)/(\gamma - 1), \quad v_i = v_j, \quad (i, j) \in \{(1, 2), (3, 4)\}, \\ v_i - v_j &= 2(c_i - c_j)/(\gamma - 1), \quad u_i = u_j, \quad (i, j) \in \{(2, 3), (4, 1)\}. \end{aligned} \tag{2}$$

For simplicity, we assume $\rho_2 = \rho_4$ and $u_1 = v_1 (= 0)$. Connections (2) are possible iff we have

$$2c_2 = c_1 + c_3. \tag{3}$$

For any fixed $\{\rho_1, p_1, u_1, v_1, \rho_3\}$, we find c_2 from compatibility condition (3) and other variables from (2) and the symmetry. We use $\gamma = 1.4$. We draw both families of (pseudo) characteristic curves corresponding to $\lambda_\pm$ by (see e.g. [7]),

$$\frac{d\eta}{d\xi} = \lambda_\pm(\xi, \eta) \equiv \frac{(u - \xi)(v - \eta) \pm c[(u - \xi)^2 + (v - \eta)^2 - c^2]^{1/2}}{(u - \xi)^2 - c^2}, \tag{4}$$

where $\xi = x/t, \eta = y/t$. The pseudo-Mach number is $M = [(u-\xi)^2 + (v-\eta)^2]^{1/2}/c$. In numerical simulation in which $\rho_1 = 1.0, p_1 = 0.444, u_1 = v_1 = 0.0, \rho_3 = 0.15, \Delta x = \Delta y = 1/3200$, shock formation is found in paper [3] which has not been expected or seen in earlier work, see Figure 1.

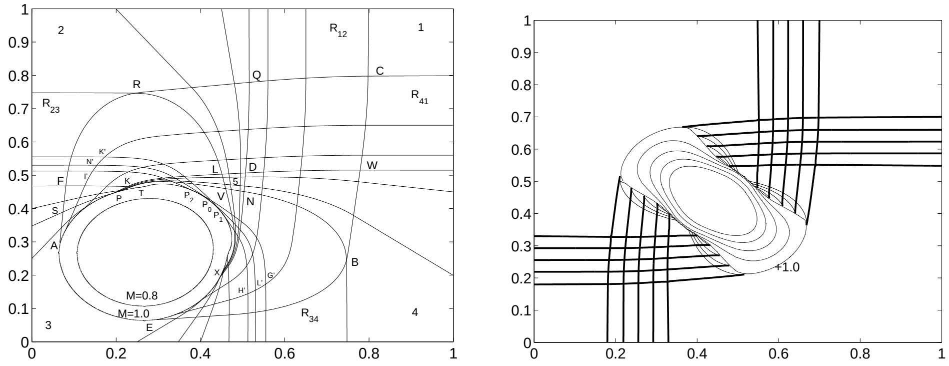

Figure 1. Left: Configuration A from [3] shows some pseudo-characteristic curves (light) and Mach number contours (bold) marked with $M = 1.0$ and $M = 0.8$, in which a shock occurs along curve AP. Right: Configuration B from [3] shows contour curves of pseudo-Mach number (light closed curves), some pseudo-characteristic curves (bold, short, and light curves), and shocks with large data.

In Config. B, the initial data (p_i, ρ_i, u_i, v_i) in the i-th quadrants $(i = 1, 2, 3, 4)$ are such that states 1 and 2 form a forward rarefaction wave R^+_{12}, states 2 and 3 form a backward rarefaction wave R^-_{23}, states 3 and 4 form a forward rarefaction wave R^+_{34}, and states 4 and 1 form a backward rarefaction wave R^-_{41}. These requirements on the data force the speed of sound to satisfy $c_2 = c_4, c_1 = c_3$, and $S_i = S_1 (i = 2, 3, 4)$. For data $p_1 = 0.444, \rho_1 = 1.0$, $u_1 = v_1 = 0.00$, $\rho_2 = 0.5197, \gamma = 1.4$, shock formation occurs, see Fig. 1.

The difficulty to a rigorous proof lies in the shortage of effective methods of analysis. Using methods that we have developed in recent years [8, 2, 6], we are able to construct a class of analytic solutions to Configuration B. We have

Theorem ([13]) *Consider Config. B for system (1). Let* $\gamma > 1 + \sqrt{2}$*. Then, there exists a number* $c^*_2(\gamma) \in (0, 1)$ *such that Config. B has a global continuous solution, provided* $0 < c_2 < c^*_2(\gamma) c_1$*. The solution has a vacuum at the center.*

Additionally, if the waves R^+_{12}, R^-_{23} etc. are not large, then shock waves form internally for Config. B. In our paper [9] we construct solutions for Config. B that show shock wave formation and other waves which we call semi-hyperbolic wave patches. For the pressure gradient system, which is a very interesting model of the Euler system, the semi-hyperbolic wave patches are constructed in paper [11].

For future simulations of Config. B, we suggest to use normalization $c_1 = 1$, and the only two free parameters are $c_2 \in (0, c_1)$ and $\gamma > 1$. The other variables are given by $u_1 = (c_1 - c_2)/(\gamma - 1) > 0,\ v_1 = v_2 = u_4 = u_1,\ u_2 = u_3 = v_3 = v_4 = -u_1,\ c_3 = c_1,\ c_4 = c_2$ with constant entropy $S_i = S_1 (i = 2, 3, 4)$. The set-up is symmetric w.r.t. both lines $\xi \pm \eta = 0$.

We hope that the properties of these solutions of Config. A and B are useful in applying these problems as testing cases for various numerical schemes.

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