Weakly Nonlinear Asymptotic Mod
of the Multi-D Compressible Euler Equ

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Abstract: We derive the unsteady transonic small
equation (UTSD) and the pressure-gradient equatio
Euler system, and comment on their developments
on (regular reflection and) doable problems.

1. Introduction

There has been remarkable progress in the study
compressible Euler equations in multi-dimensions, se
of Shuxing Chen , Zhouping Xin and Huicheng Yin
Zhang, and Guiqiang Chen and Feldman. There are a
simplifications of the Euler system such as the isentro
the irrotational (potential) flow equations, or steady f
we want to present some asymptotic models: Th
unsteady transonic small disturbance equation (UTS
pressure gradient equation. They offer different pers
the full Euler system.

The full Euler system for an ideal fluid takes the fo

$$\begin{aligned}&\rho_t + \nabla \cdot (\rho u) = 0,\\ &(\rho u)_t + \nabla \cdot (\rho u \otimes u + pI) = 0,\\ &(\rho E)_t + \nabla \cdot (\rho E u + pu) = 0,\end{aligned}$$

where

$$E := \frac{1}{2}|u|^2 + e$$

where e is the internal energy. For a polytropic gas,

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

where $\gamma > 1$ is the gas constant.

2. Pressure gradient system

2.1. The asymptotic derivation

We consider an ideal polytropic gas, therefore

$$E = \frac{1}{2}|u|^2 + \varepsilon p/\rho$$

where

$$\varepsilon := \frac{1}{\gamma - 1}.$$

We look for an asymptotic solution of the form

$$\begin{aligned} \rho &= \rho_0 &+ \quad &\varepsilon\rho_1 + O(\varepsilon^2), \\ u &= & &\varepsilon u_1 + O(\varepsilon^2), \\ p &= & &\varepsilon p_1 + O(\varepsilon^2). \end{aligned}$$

This scaling corresponds to sound speeds of the ord

$$c = \sqrt{\gamma p/\rho} = O(\varepsilon^0).$$

So we scale space and time variables by the same fa
$O(1)$) to study acoustic phenomena. [*]

The leading order perturbation equation from cons
mass is

$$(\rho_0)_t = 0,$$

so

$$\rho_0 = \rho_0(x).$$

The leading order equation from conservation of r
$O(\varepsilon)$ and conservation of energy $O(\varepsilon^2)$ form a varia

[*] This is called order one, but it is the zeroth exponent term ε

pressure-gradient system

$$(\rho_0 u_1)_t + \nabla p_1 = 0,$$

$$\left(\tfrac{1}{2}\rho_0 |u_1|^2 + p_1\right)_t + \nabla \cdot (p_1 u_1) = 0.$$

For smooth solutions, the energy equation can be re

$$(p_1)_t + p_1 \nabla \cdot u_1 = 0.$$

One can eliminate u_1 to form a single equation for

$$\left(\frac{(p_1)_t}{p_1}\right)_t - \nabla \cdot \left(\frac{1}{\rho_0} \nabla p_1\right) = 0.$$

If we let $\rho_0 = 1$, then we obtain a nice equation

$$\left(\frac{(p_1)_t}{p_1}\right)_t - \Delta p_1 = 0.$$

The above derivation was from John Hunter via pr
munication. This asymptotic regime lacks strong phy
because there is no such physical material for ve
There is no other nonphysical concerns though.
ple, it is perfectly ok to have $p = \varepsilon p_1 + O(\varepsilon^2) \rightarrow$ 0
an independent variable and one can adjust the temp
achieve that.

One guiding principle in asymptotics is this philos
can look at a physical process at any scale one wi
point is whether you find anything interesting there.

Wake-up question: Can one shift the pressure by
without affecting the equation?

2.2. The flux-splitting theme

We consider a system of ordinary differential equat

$$\frac{d\mathbf{u}}{dt} - (\mathbf{A} + \mathbf{B})\mathbf{u} = \mathbf{0}, \quad (t > 0)$$

with initial data

$$\mathbf{u}(0) = \mathbf{u}_0,$$

where $\mathbf{u}$ and $\mathbf{u}_0$ are vectors in $\mathbb{R}^n$ and $\mathbf{A}$ and $\mathbf{B}$ are $n\times$
A splitting idea is to split the sum $\mathbf{A} + \mathbf{B}$ into two ar
the two separate systems

$$\frac{d\mathbf{v}}{dt} - \mathbf{A}\mathbf{v} = \mathbf{0},$$

and

$$\frac{d\mathbf{w}}{dt} - \mathbf{B}\mathbf{w} = \mathbf{0}.$$

Use the original data

$$\mathbf{v}(0) = \mathbf{u}_0$$

for (7). Use the value $\mathbf{v}(\Delta t)$ of the solution $\mathbf{v}(t)$ at t
as initial data for (8):

$$\mathbf{w}(0) = \mathbf{v}(\Delta t).$$

The hope of the idea is that the solution $\mathbf{w}(t)$ at ti
$\mathbf{w}(\Delta t)$, is the same or close to the original solution $\mathbf{u}$
Δt is small.

We can quickly determine the merit of the idea. T
to (5) (6) can be written as

$$\mathbf{u}(\Delta t) = e^{(\mathbf{A}+\mathbf{B})\Delta t}\mathbf{u}_0.$$

The final value of $\mathbf{w}(\Delta t)$ is

$$\mathbf{w}(\Delta t) = e^{\mathbf{B}\Delta t}\mathbf{v}(\Delta t) = e^{\mathbf{B}\Delta t}e^{\mathbf{A}\Delta t}\mathbf{u}_0.$$

It is clear that $\mathbf{u}(\Delta t) = \mathbf{w}(\Delta t)$ for all Δt if $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{AB} = \mathbf{BA}.$$

In general (i.e., commutative or not), we obtain

$$\begin{aligned} \mathbf{u}(\Delta t) &= \mathbf{w}(\Delta t) + O\left((\Delta t)^2\right) \\ &= \mathbf{u}_0 + \Delta t(\mathbf{A} + \mathbf{B})\mathbf{u}_0 + O\left((\Delta t)^2\right) \end{aligned}$$

for small Δt.

To get to time $t = 2\Delta t$, one uses $\mathbf{w}(\Delta t)$ as the
for $\mathbf{v}$ and start the same process again. The purpo
ting an equation is to achieve two problems which
suitable for computation. The philosophical backbo
the simultaneous evolution process by $\mathbf{A} + \mathbf{B}$ can b
approximately by the individual processes $\mathbf{A}$ and $\mathbf{B}$ e
ternatively in equal short time intervals. A trivial exa
consider 1-D Euler in x, y, z directions.

2.3. The splitting derivation

The pressure gradient system was first obtained fro
splitting scheme of Agarwal and Halt (1994). It was
justified for small velocity and large γ values in Zhe

The philosophy is the observation that fluid flows
cause it has been flowing (inertia) or because there is
difference (pressure gradient). Separating the pressu
inertia in the flux of the Euler equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u E + u p \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v E + v p \end{pmatrix}_y = 0$$

where

$$E = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1}\frac{p}{\rho},$$

we obtain two systems of equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho uv \\ \rho uE \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 \\ \rho vE \end{pmatrix}_y = 0$$

and

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} 0 \\ p \\ 0 \\ up \end{pmatrix}_x + \begin{pmatrix} 0 \\ 0 \\ p \\ vp \end{pmatrix}_y = 0\,.$$

Agarwal and Halt have used this splitting (11)(12)
novel scheme in numerical computations for airfoil
observed a consistent improvement over other sche
AUSM, CUSP, and Van Leer). Systems (11) and (1
called the *transport* (or *convective*) and the *pressu*

systems respectively. Thus this splitting corresponds
aration of the two mechanisms– pressure difference a
responsible for fluid motion.

We do not deal with (11) in this talk. Right now w
interested in system (12), which resembles (10) more
does in their essential nonlinear structures. We sir
slightly. From the first equation of system (12) we

$$\rho_t = 0.$$

Thus ρ is independent of time. For simplicity, we wi

$$\rho = 1.$$

Then system (12) can be written as

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ E_t + (up)_x + (vp)_y = 0, \end{cases}$$

where $E = (u^2+v^2)/2+p/(\gamma-1)$. For smooth solutio gions where a solution is smooth, system (13) can b to be

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ \frac{1}{\gamma-1} p_t + p u_x + p v_y = 0. \end{cases}$$

Through the transformation

$$\begin{cases} p = (\gamma - 1)P, \\ t = \frac{1}{\gamma-1} T, \end{cases}$$

system (14) can be rewritten to be

$$\begin{cases} u_T + P_x = 0, \\ v_T + P_y = 0, \\ P_T + P u_x + P v_y = 0. \end{cases}$$

From system (16) we can find that

$$\left(\frac{P_T}{P}\right)_T = P_{xx} + P_{yy}$$

which we think is a very interesting, and one of th
2-D quasilinear wave equation.

2.4. Progress of research

Both Cauchy and Riemann problems for systems
or (17) are open.

- Peng Zhang, Jiequan Li, and Tong Zhang (D
given a set of conjectures (with numerics) for solut
four-constant four-wave Riemann problem for thes
see Li Jiequan et al's or Section 9.3 of Zheng's boo

The self-similar coordinates

$$\xi = \frac{x}{T}, \quad \eta = \frac{y}{T}$$

can reduce the Riemann problem by one dimension
all three systems (12), (13), and (17), and even thei
versions are of mixed type in the self-similar coordir
major difficulty in proving the conjectures is this char
combined with the nonlinearity of the systems.

- Zheng (CPDE, 1997) has established the existe
lutions in the elliptic region. Equation (17) in the
coordinates (ξ, η) takes the form

$$\begin{aligned}(P-\xi^2)P_{\xi\xi} &-2\xi\eta P_{\xi\eta} + (P-\eta^2)P_{\eta\eta} \\ &+\tfrac{1}{P}(\xi P_\xi + \eta P_\eta)^2 - 2(\xi P_\xi + \eta P_\eta) =\end{aligned}$$

The eigenvalues of the coefficient matrix of the se
terms of (18) can be found to be P and $P-\xi^2-$

proved in (cpde, 1997) the existence of a weak s
equation (18) in any open, bounded and convex reg
with smooth boundary and the degenerate boundary

$$P|_{\partial\Omega} = \xi^2 + \eta^2$$

provided that the boundary of Ω does not contain
$(0,0)$.

- Kyungwoo Song (2003, CPDE (in press) has re
restriction on the origin and the complete smoothn
boundary. • Kim and Song (preprint 2003) have obt
larity of the solution in the interior of the domain and
up to and includes the boundary.

- Dai and Zhang (ARMA, 2000) have obtained the
of two rarefaction waves adjacent to the vacuum.

- Zheng has recently established the existence o solution involving a shock as a free boundary (AMA press)).

Jiequan Li has been able to use the result as mo solve the gas expansion to vacuum problem for the system.

3. UTSD equation

Let the strength of incident shock be ϵ, the angle o be $\alpha = O(1)\sqrt{\epsilon}$, and $c_0 = \sqrt{\gamma p_0/\rho_0}$ be the sound spe on a region of the triple point. See a Figure.

Scale the independent variables

$$\hat{x} = \frac{x - c_0 t}{\epsilon}, \qquad \hat{y} = \frac{y}{\sqrt{\epsilon}}, \qquad \hat{t} = t.$$

Scale the dependent variables

$$\begin{pmatrix} \rho \\ \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \rho_0 \\ 0 \\ p_0 \end{pmatrix} + \epsilon \begin{pmatrix} \rho^{(1)} \\ \mathbf{u}^{(1)} \\ p^{(1)} \end{pmatrix} + \epsilon^{3/2} \begin{pmatrix} \rho^{(2)} \\ \mathbf{u}^{(2)} \\ p^{(2)} \end{pmatrix} + O($$

It turns out that

$$\begin{pmatrix} \rho^{(1)} \\ \mathbf{u}^{(1)} \\ p^{(1)} \end{pmatrix} = \widehat{u} \begin{pmatrix} \rho_0 \\ c_0\mathbf{e}_{\widehat{x}} \\ \gamma p_0 \end{pmatrix}, \qquad \begin{pmatrix} \rho^{(2)} \\ \mathbf{u}^{(2)} \\ p^{(2)} \end{pmatrix} = \widehat{v} \left(\begin{array}{l} 0 \\ c_0\mathbf{e}_{\widehat{y}} \\ 0 \end{array} \right.$$

The equation is

$$\begin{cases} \widehat{u}_{\widehat{t}} + (\frac{c_0}{2\sigma}\widehat{u}^2)_{\widehat{x}} + \frac{1}{2}c_0\widehat{v}_{\widehat{y}} & = 0, \\ \widehat{v}_{\widehat{x}} - \widehat{u}_{\widehat{y}} & = 0 \end{cases}$$

where $\sigma = 2/(\gamma + 1)$. Using scaling, one obtain the

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x + v_y & = 0 \\ v_x - u_y & = 0. \end{cases}$$

This presentation is from Chapter 11 of Zheng's b

4. Non-asymptotic models

For comparison, we list some non-asymptotic mod

- We can assume that the entropy $S = Ap/\rho^{\gamma}$ is a c that the Euler system has one equation fewer. This i *isentropic Euler system*, consisting of the equations

- We can assume that the flow initially contains n and it follows that the flow will have no vorticity for l thus resulting a irrotational flow system, generally as *potential flows*.

5. Typical problems

One proposes initial value or boundary value, or a c of both values problems.

Riemann problems have physical background in 1-

ficial for 2-D.

Mach reflection problems (oblique shock reflection

are set in 2-D and are a constant source of open pr

Flow in a channel with a bump, such as considered

Cook (book, 1986), is another interesting physical si

transonic problems.

Summary

We have derived the pressure-gradient and the UTS