

A global solution to a two-dimensional Riemann problem involving shocks as free boundaries

Yuxi Zheng¹

Department of Mathematics
The Pennsylvania State University
University Park, PA 16802

Abstract

We present a *global* solution to a Riemann problem for the pressure gradient system of equations. The Riemann problem has initially two shock waves and two contact discontinuities. The angle between the two shock waves is set initially to be close to 180 degrees. The solution has a shock wave that is usually regarded as a free boundary in the self-similar variable plane. Our main contribution in methodology is handling the tangential oblique derivative boundary values.

Keywords: Pressure gradient equation, free boundary, shock waves, tangential oblique derivative, 2-D Riemann problem, gas dynamics.

AMS subject classification: Primary: 35L65, 35J70, 35R35; Secondary: 35J65.

1 Introduction

Recent important work of Canic, Keyfitz, Lieberman, and Kim on the problem of existence of solutions to the transonic small disturbance system of conservation laws (TSD) [5][6][7] promise further development. In this paper, we try their approach on the pressure gradient system of equations. The advantage of this system is that it has similar problems as the 2-D steady or unsteady TSD equations but with bounded subsonic domains instead of unbounded subsonic domains. We are able to carry their approach further to finally obtain *global* solutions. The new ingredient is the handling of loss of uniform obliqueness.

¹Research partially supported by NSF-DMS-0071858, 0305497, 0305114

We refer the reader to references Zheng [30], Dai and Zhang [11], Song [27], Kim and Song [16], Zheng's book [31], and Li, Zhang and Yang's book [17] for background information and recent progress regarding the pressure gradient system. For recent related work on multi-dimensional shock free boundaries, we refer to Chen and Feldman [8][9] and Shuxing Chen [10] in addition to the work [5][6][7].

The proof of the result is based heavily on the aforementioned work [5][6][7]. A full and self-contained presentation of the proof will be lengthy and repetitive. We present in this write-up the result and the portion of the proof that is needed in addition to the work done in [5][6][7]. We apologize to the nonexpert readers that the presentation is not self-contained.

2 The gist

The pressure gradient system takes the form

$$\begin{cases} u_t + p_x = 0 \\ v_t + p_y = 0 \\ E_t + (up)_x + (vp)_y = 0, \end{cases} \quad (1)$$

where $E = (u^2 + v^2)/2 + p$. Again, we refer the reader to the books [31][17] for background information. The system appeared first in Agarwal and Halt [1]. Roughly speaking, this system is the Euler system in gas dynamics without the inertial parts.

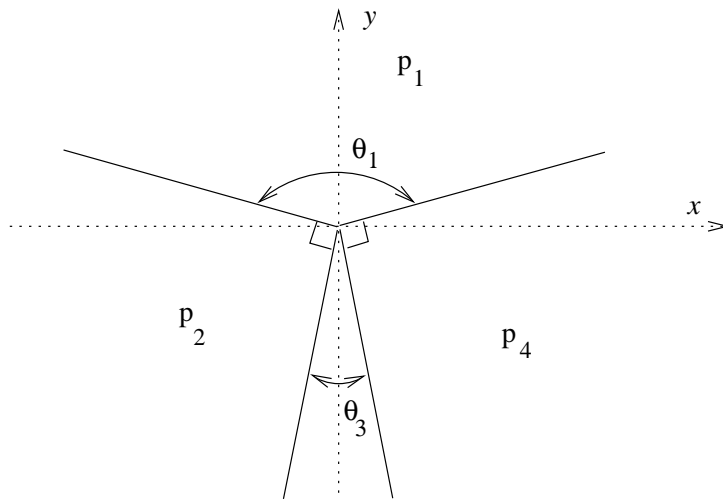


Figure 1. The four sectors in the initial data.

The Riemann problem for (1) is an initial value problem. The initial values are such that they are independent of the variable r , where (r, θ) denote the polar coordinate system in the (x, y) plane. This is very general. A special case is when the initial data

are constant in each of the four quadrants. In this paper, however, we deal with a type of data that are constant in four sectorial regions with sectorial angles $\theta_1 \approx \pi, \theta_2 = \pi/2, \theta_3 \approx 0, \theta_4 = \pi/2$. See Figure 1. As we will see, the condition $\theta_2 = \pi/2, \theta_4 = \pi/2$ can be relaxed to other angles easily.

We further require the following four conditions on the initial data.

- 1) A single (forward) shock will form from the one-dimensional interaction between the states 1 and 4. The shock is denoted S_{14}^+ .
- 2) A single (backward) shock will form from the one-dimensional interaction between the states 1 and 2. The shock is denoted S_{12}^- .
- 3) A contact discontinuity forms from the one-dimensional interaction between the states 2 and 3. The contact discontinuity is denoted J_{23}^+ .
- 4) Another contact discontinuity forms from the one-dimensional interaction between the states 3 and 4. The contact discontinuity is denoted J_{34}^- .

This type of data exists, see Configuration H of the book of Li et al [17]. Thus, in the self-similar plane $(\xi, \eta) = (x/t, y/t)$, the solution will look like Figure 2 before we settle the interaction of these four waves within the sonic circle of the state 1: $\xi^2 + \eta^2 = p_1$.

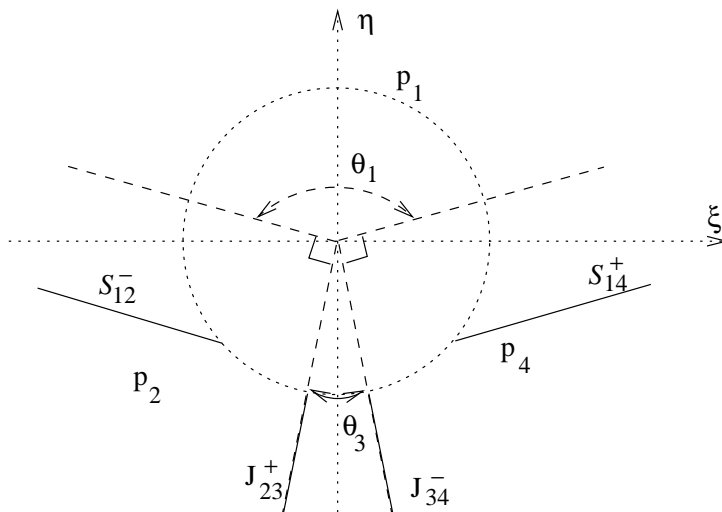


Figure 2. The far-field solution at $t=1$.

In this paper we prove that the solution will be like in Figure 3, where the two shocks bend slightly and meet to form a smooth curve, provided that $p_1 > p_2 = p_3 = p_4$ and the angle θ_1 is close to π . The two contact discontinuities remain contact discontinuities and vanish at the origin. The pressure p is C^2 smooth in the entire subsonic domain. The condition that θ_1 is close to π makes this problem a perturbation of a single 1-d shock wave.

Main Theorem. There exists an (entropy) solution (p, u, v) defined for all $(\xi, \eta) \in \mathbb{R}^2$, provided that $p_1 > p_2 = p_3 = p_4$ and the angle θ_1 is close to π . The shock curve is C^1 smooth. The pressure p is C^2 smooth in the subsonic domain.

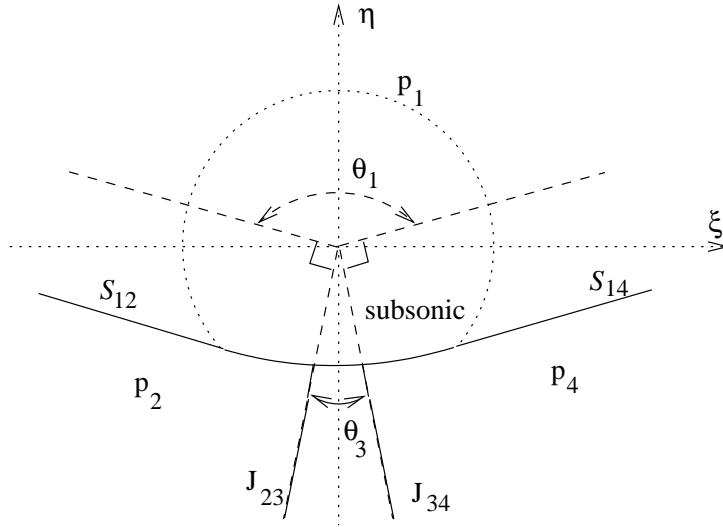


Figure 3. The solution.

The difficulty in this paper is the degeneracy at the point at which the two shocks meet, as the two shocks actually belong to two different classes, and the obliqueness fails (see later). In fact, the directions of the oblique derivatives are tangent to the boundary at the middle point, see Figure 4, and the degeneracy type of the boundary value is known as *emergent* type in the linear case, see [23]. We use a cut-off to remove the degeneracy and later remove the cutoff, which has been attempted in Canic, Keyfitz, and Kim in [6][7]. The significance is that we can remove this cut-off and this solution is thus global.

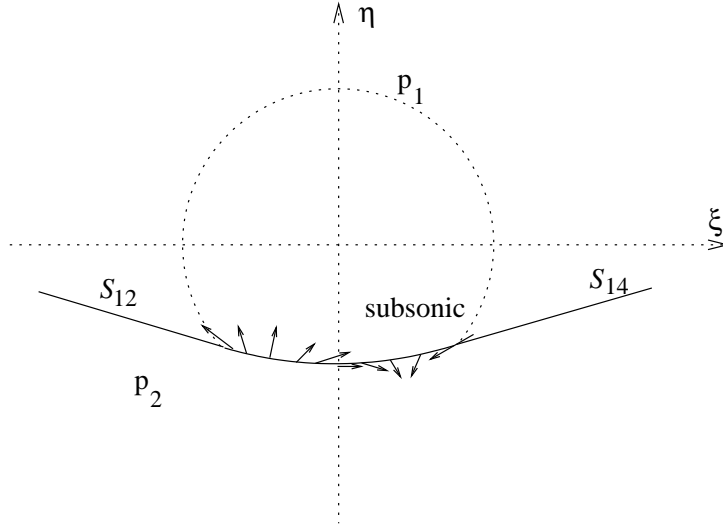


Figure 4. Directions of the tangential oblique derivatives.

3 Technical formulation

To be more precise, the Riemann problem for (1) is an initial value problem of the form

$$(u, v, p)|_{t=0} = (u_i, v_i, p_i) \quad \text{in sector } i, \quad i = 1, 2, 3, 4. \quad (2)$$

In the self-similar plane, the system of equations are

$$\begin{cases} -\xi u_\xi - \eta u_\eta + p_\xi = 0, \\ -\xi v_\xi - \eta v_\eta + p_\eta = 0, \\ -\xi E_\xi - \eta E_\eta + (pu)_\xi + (pv)_\eta = 0. \end{cases} \quad (3)$$

The initial condition (2) becomes boundary condition

$$\lim(u, v, p) = (u_i, v_i, p_i) \quad \text{in sector } i, \quad i = 1, 2, 3, 4 \quad (4)$$

as the self-similar radius $(\xi^2 + \eta^2)^{1/2}$ approaches infinity along a fixed ray.

Across a shock, the Rankine-Hugoniot relation is

$$\frac{d\eta}{d\xi} = \sigma_\pm = \frac{\xi\eta \pm \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\xi^2 - \bar{p}}, \quad (5)$$

$$[u] = \frac{\xi \bar{p} \pm \eta \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\bar{p}(\xi^2 + \eta^2)} [p], \quad (6)$$

$$[v] = \frac{\eta \bar{p} \mp \xi \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\bar{p}(\xi^2 + \eta^2)} [p]. \quad (7)$$

Again, we refer the reader to the books [31][17] for the derivation of these relations and other technical informations below.

One eliminates the (u, v) variables in system (3) to derive a de-coupled equation for p in smooth regions:

$$(p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \quad (8)$$

Once p is found, we use the first two equations in (3) and the inner boundary values given by (6-7) to evaluate (u, v) . So we shall focus on (8).

A nontrivial issue is to make sure the two systems are equivalent. We will impose boundary conditions on (8) to guarantee the equivalence. That is, the oblique derivative boundary condition will suffice provided that the free boundary does not contain perfect circular arcs.

We take an angle $\theta_1 \in (0, \pi)$ for the angle of the first sector and a state (u_1, v_1, p_1) for the first sector. We take a pressure p_2 in the interval $(0, p_1)$ for the pressure in the other three sectors. We will take $p_3 = p_4 = p_2$ in the remainder of the paper. Now we imagine an open bounded set $\Omega \subset \mathbb{R}^2$, whose boundary $\partial\Omega$ consists of two simple parts: σ and Σ , where σ is an arc of a sonic circle and Σ is a couple of shock waves: Σ^+ and Σ^- . See Figure 5, where

$$\eta_{m2} := -\sqrt{(p_m + p_2)/2} \quad (9)$$

and p_m is an internal pressure between p_1 and p_2 .

We notice that the angle $\theta_1 \in (0, \pi)$ is to control the slope of Σ . Alternatively, in place of $\theta_1 \in (0, \pi)$, we can use the $p_m \in (p_2, p_1)$ to do the same thing. Let Σ be a curve starting at the point $(\xi, \eta) = (0, \eta_{m2})$. When p_m is close to p_1 , we have θ_1 close to π (proved later).

An alternative derivation of the oblique derivative condition is as follows. The second-order equation for p plus the two first-order equations for (u, v) implies

$$(\xi\partial_\xi + \eta\partial_\eta)L + L = 0, \quad (12)$$

for

$$L := (\xi\partial_\xi p + \eta\partial_\eta p)/p - u_\xi - v_\eta. \quad (13)$$

This ODE (12) in the form $r\partial_r L + L = 0$ has the only solution $L = 0$ if $L(r_0) = 0$ at any point $r = r_0 > 0$. Thus the third equation of (3) will hold if it holds on the boundary Σ . We then use integral representations to represent (u, v) through p and insert them into $L = 0$ on Σ and simplify the expression to yield (10).

Free boundary problem: *Given three pressure values $p_2 < p_m < p_1$. Find a curve Σ and a function p such that p satisfies equation (8) with Dirichlet condition*

$$p = p_1, \text{ on } \sigma : \xi^2 + \eta^2 = p_1, (\theta_B \leq \theta \leq \theta_A) \quad (14)$$

and oblique derivative condition (10) on

$$\Sigma : \eta = \eta(\xi) \quad \text{defined by } \eta'(\xi) = \sigma_\pm, \eta(0) = \eta_{m2} \quad (15)$$

where σ_\pm are given by the first R-H relation (5). See Fig. 5.

We approach the above free boundary problem through a modified problem which we call a *naturally modified problem*. See Figure 6, where $\eta_{12} := -\sqrt{(p_1 + p_2)/2}$.

Naturally modified problem: *Given $\epsilon_e > 0$ and $\epsilon_D > 0$ and three pressure values $p_2 < p_m < p_1$. Solve (p, Σ) from the equation*

$$\epsilon_e \Delta p + (p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \quad (16)$$

modified from equation (8), with Dirichlet condition

$$\begin{aligned} p &= p_1, & \text{on } \sigma : \xi^2 + \eta^2 &= p_1, (\theta_B \leq \theta \leq \theta_A) \\ p &= p(D) := p_m + \frac{1}{4}\epsilon_D^2 & \text{on } \sigma : \eta &= \eta_{m2}, \xi \in [-\epsilon_D, \epsilon_D] \end{aligned} \quad (17)$$

and oblique derivative condition (10) on

$$\Sigma : \eta = \eta(\xi) \quad \text{defined by } \eta'(\xi) = \sigma_\pm, \eta(\pm\epsilon_D) = \eta_{m2} \quad (18)$$

where (ϵ_e, ϵ_D) will be sent to zero, and σ_\pm are given by the first R-H relation (5). See Fig. 6.

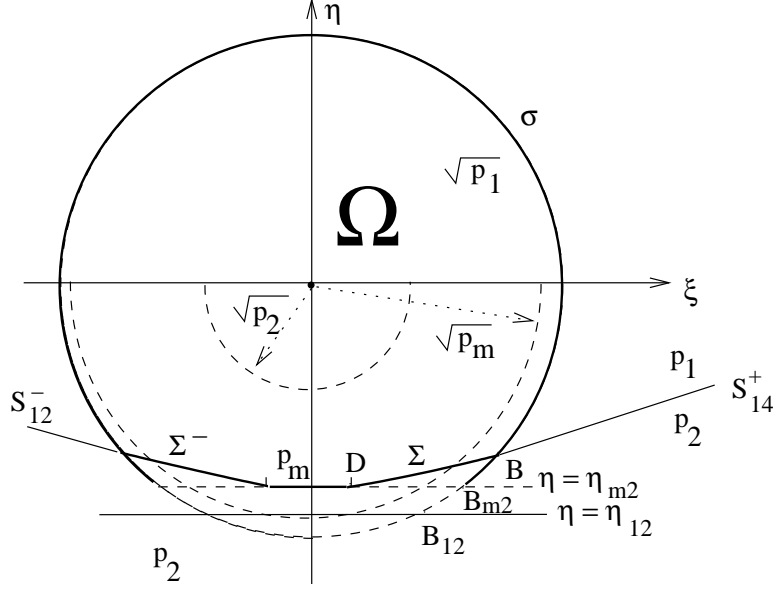


Figure 6. Naturally modified mixed Dirichlet and oblique derivative boundary value problem.

We will work on a variation of the above Naturally Modified Problem and call it the *Core Problem*, see below. We realize that we have many options in removing the degeneracy at the point $(\xi, \eta) = (0, \eta_{m2})$. We also have many options for the smoothness of an approximation of Σ near the point D . The set-up in the above Naturally Modified Problem is only one intuitive way, which introduces a corner D . To use Lieberman's result on optimal regularity [20], we find out to our surprise that the regularity of the solution is better if the corner D has a sharper (more acute) angle so as to make the direction at the corner of the oblique vector in the oblique derivative condition point from the *exterior to the exterior* of the domain. Thus, in place of the straight line segment $\eta = \eta_{m2}$, $\xi \in [-\epsilon_D, \epsilon_D]$ we use a circular arc symmetric with respect to the η -axis.

Let $\eta = A(\xi)$ be a circular arc centered at a point $(\xi, \eta) = (0, \eta_{\epsilon_D})$ with radius $r_{\epsilon_D} > 0$ for $\xi \in [-\epsilon_D, \epsilon_D]$. We will take $\eta_{\epsilon_D} = 0$ although we can take it to be either negative or positive with absolute value in the order of ϵ_D or larger. We take $r_{\epsilon_D} = \epsilon_D$ or accordingly.

Core problem: Given $\epsilon_e > 0$ and $\epsilon_D > 0$ and three pressure values $p_2 < p_m < p_1$. Solve (p, Σ) from the equation

$$\epsilon_e \Delta p + (p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \quad (19)$$

modified from equation (8), with Dirichlet condition

$$\begin{aligned} p &= p_1, & \text{on } \sigma : \xi^2 + \eta^2 &= p_1, (\theta_B \leq \theta \leq \theta_A) \\ p &= p(D) := p_m + \frac{1}{4}\epsilon_D^2 & \text{on } \sigma : \eta &= A(\xi), \xi \in [-\epsilon_D, \epsilon_D] \end{aligned} \quad (20)$$

and oblique derivative condition (10) on

$$\Sigma : \eta = \eta(\xi) \quad \text{defined by } \eta'(\xi) = \sigma_{\pm}, \eta(\pm\epsilon_D) = \eta_{m2} \quad (21)$$

where (ϵ_e, ϵ_D) will be sent to zero, and σ_{\pm} are given by the first R-H relation (5). See Fig. 7.

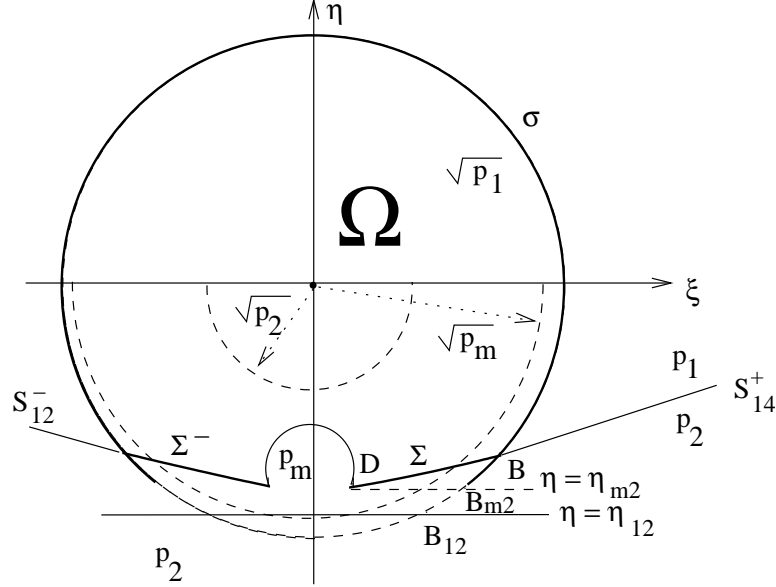


Figure 7. Core problem: Modified mixed Dirichlet and oblique derivative boundary conditions.

One might think that an alternative of the circular symmetric arc would be a smooth piece of curve which connects the rest of Σ smoothly at D . In this scenario, we have a mixed Dirichlet and oblique derivative boundary condition on a smooth curve. Unfortunately, there is no written English work on this problem, a big surprise, although we find that there are claims that the regularity can be established with classically known methods.

To utilize the available English references (rather than hard-to-find former USSR papers) on the regularity of solutions at corners, we will use the circular arc as approximation. For those who are interested in a smooth approximation rather than this odd arc approximation, we refer to a two-page announcement of Kerimov [18] (courtesy of Lieberman).

Also, we will see that there are quite a few papers on tangential oblique derivative problems on linear problems, starting with Hörmander [15], Egorov and Kondrat'ev [12], and many afterward ([2][3][4][21][22][23][24][25][26][28][29]), culminates in Guan and Sawyer [14], but none of them applies directly to our problem, albeit Guan and Sawyer's [14] is extremely close. We point out that the book of Gilbarg and Trudinger [13] does not deal with tangential oblique derivative problems. We choose to work out

our problem rather than fit our problem into their frame in this paper. Future work may well utilize these papers.

It can be proved that a solution (p, Σ) to the Free Boundary Problem yields a solution to the original Riemann problem (1)(2) provided that the shock wave Σ does not contain any arc of circle, which is the case for our current problem in the limits $\epsilon_e \rightarrow 0, \epsilon_D \rightarrow 0$.

We remark that in the work [5][6][7] this type of problem is approached through working at the point B . Our approach is to work at the point D . We point out that we have checked that contact discontinuities can be handled easily.

4 Proof

We present the novel part of the proof in detail. The basic steps of the proof which are similar to [5][6][7] are sketched only.

4.1 A priori estimates

We modify the main equation (19) in the Core Problem. First the denominator p can be replaced by $N(p)$ where N is C^3 smooth in \mathbb{R}^1 bounded from above and below by two positive numbers and is the identity function in the interval $[p_m, p_1]$. Second, replace the two simple p 's in (19) by $K(\xi, \eta, p)$ where

$$K(\xi, \eta, z) = \begin{cases} z, & \text{if } z \geq \xi^2 + \eta^2, \\ \xi^2 + \eta^2, & \text{if } z < \xi^2 + \eta^2. \end{cases} \quad (22)$$

This K is Lipschitz in \mathbb{R}^3 .

We modify the oblique derivative boundary condition. We use the new form

$$\sigma_{\pm} = \frac{\bar{p} - \eta^2}{\xi(-\eta) \pm \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}} \quad (23)$$

for σ_{\pm} from the first R-H formula. All the p 's in σ_{\pm} and the oblique derivative condition are replaced by $N(p)$, except the terms (p_{ξ}, p_{η}) . This is the third cut-off: Replace all appearances of the term $\xi^2 + \eta^2 - \bar{p}$ by $\varphi(\xi^2 + \eta^2 - \bar{p})$ where $\varphi(z)$ is smooth in \mathbb{R}^1 , equals to the identity for $z > \frac{1}{2}\epsilon_D^2$, and is otherwise smooth and bounded below by $\frac{1}{4}\epsilon_D^2$. The cut-off level $\frac{1}{2}\epsilon_D^2$ is lower than the value of $\xi^2 + \eta^2 - \bar{p}$ at point D, which is $\frac{7}{8}\epsilon_D^2$.

We list some hard facts:

$$\xi_{B_{12}}^2 = \frac{1}{2}(p_1 - p_2), \quad \xi_{B_{m2}}^2 = \frac{1}{2}(p_1 - p_2) + \frac{1}{2}(p_1 - p_m). \quad (24)$$

The parameters are ordered to be

$$\epsilon_D < \epsilon_e < p_1 - p_m. \quad (25)$$

Lemma 4.1 (*Maximum principle*) *Any C^2 solution p of the core problem satisfies the bounds*

$$p_m < p(D) \leq p \leq p_1 \quad (26)$$

provided that $|\sigma_{\pm}| < (p_m - p_2)/(2p_1)$.

Proof. We need p be $C(\bar{\Omega})$, $C^1(\Omega \cup \Sigma)$, and $C^2(\Omega)$, where Ω is open. We need Σ be in $H_{1+\alpha}$. These smoothness will be justified.

The equation is uniformly elliptic, and the coefficients are at least Lipschitz. It is well-known that there is no extreme values in the interior.

We show that there is no extreme values on Σ . This is done by Hopf's lemma and the oblique derivative condition.

Obliqueness: An interior normal of Σ is $(-\sigma_{\pm}, 1)$. The obliqueness is defined by the inner product of a unit (interior) normal with the oblique direction $(-\sigma_{\pm}, 1) + \frac{\xi\sigma_{\pm} - \eta}{p}(\xi, \eta)$. We ignore the tangential direction. We ignore the normalization of the interior normal for now as well. Thus

$$\text{Obliqueness} = (-\sigma_{\pm}, 1) \cdot \left\{ (-\sigma_{\pm}, 1) + \frac{\xi\sigma_{\pm} - \eta}{N(p)}(\xi, \eta) \right\}. \quad (27)$$

We do some simple algebra to yield, where the modification $N(\cdot)$ is omitted,

$$\text{Obliqueness} = \frac{p - \xi^2}{p} \left\{ \left(\sigma_{\pm} + \frac{\xi\eta}{p - \xi^2} \right)^2 + \frac{p(p - \xi^2 - \eta^2)}{(p - \xi^2)^2} \right\}. \quad (28)$$

The obliqueness is obvious provided that $p - \xi^2 - \eta^2 > 0$ on Σ . We will need $\sigma_{\pm} \approx 0$ to get the obliqueness before we establish that $p - \xi^2 - \eta^2 > 0$ on Σ .

We can easily see that there can be no global maximum on Σ , since a global maximum on Σ would mean that $p \geq p_1$ and that the tangential derivative of p is zero, thus an oblique derivative would be zero due to the oblique derivative condition, but this would contradict Hopf's lemma.

To avoid a global minimum on Σ , we need σ_{\pm} small to guarantee the obliqueness. The smallness depends only on p_m, p_1 , and p_2 . We have a different form

$$\text{Obliqueness} = \frac{1}{p} \left\{ p - \eta^2 + 2\xi\eta\sigma_{\pm} + (p - \xi^2)\sigma_{\pm}^2 \right\}. \quad (29)$$

At $\sigma_{\pm} = 0$, the obliqueness is $(N(p) - \eta^2)/N(p)$ which is positive along the now flat Σ . From this obliqueness and Hopf's lemma, we conclude that there is no global minimum on Σ either.

The smallness can be found as follows. We require that

$$2\xi\eta\sigma_{\pm} + N(p) - \eta^2 > 0.$$

The minimum of $N(p)$ is taken approximately as p_m , $\eta \approx \eta_{m2}$, and $2|\xi\eta| \leq \xi^2 + \eta^2 \leq p_1$. Thus we only need

$$|\sigma_{\pm}| \leq \frac{p_m - p_2}{2p_1}. \quad (30)$$

Note that the difference is indeed $p_m - p_2$ rather than $p_m - p_1$.

The sign of $p - \xi^2$ needs attention, too. Our $N(p)$ has min p_m . Our ξ has approximately an upper bound equal to $\xi_{B_{m2}}$. So it is positive. Thus there is no global minimum or maximum of p on Σ . Hence the proof is complete. QED

We can now omit the cut-off $N(p)$.

Lemma 4.2 (*Monotonicity*) *Any C^2 solution p of the core problem and any Σ yield*

$$\sigma_+ > 0 \quad \text{on } \Sigma^+. \quad (31)$$

Proof. We use the previous lemma to obtain $p \geq p(D)$. Then we have

$$(\bar{p} - \eta^2)|_D \geq \frac{1}{2}(p_m + p_2 + \frac{1}{4}\epsilon_D^2) - \frac{1}{2}(p_m + p_2) = \frac{1}{8}\epsilon_D^2 > 0. \quad (32)$$

Using the formula (23) for σ_+ we find that the derivative $\eta'(\xi) = \sigma_+$ at $\xi = \epsilon_D$ is positive, thus Σ^+ is increasing, so $\eta(\xi)$ is increasing, thus $\bar{p} - \eta^2$ remains positive. This completes the proof. QED

Lemma 4.3 (*Interior ellipticity*) *The ellipticity holds in the interior*

$$p - \xi^2 - \eta^2 > 0, \quad \text{in } \Omega. \quad (33)$$

Proof. See Zheng [30], pp. 1857-8.

Lemma 4.4 (*Edge ellipticity*) *The ellipticity holds on the free boundary*

$$p - \xi^2 - \eta^2 > 0, \quad \text{on } \Sigma^+. \quad (34)$$

Proof. We use the notation $w := p - \xi^2 - \eta^2$. Suppose w has a nonpositive minimum on Σ^+ at a point (ξ_0, η_0) . By the previous lemma, we have

$$w = 0, w_\xi + \sigma_+ w_\eta = 0, \nabla w \cdot \nu \geq 0, \text{ at } (\xi_0, \eta_0) \in \Sigma^+ \quad (35)$$

Using $\nabla p = \nabla w + 2(\xi, \eta)$ at (ξ_0, η_0) , we rewrite the oblique derivative in terms of w :

$$\begin{aligned} & (-\sigma_+ w_\xi + w_\eta) + (\xi w_\xi + \eta w_\eta) \frac{\xi \sigma_+ - \eta}{p} \\ & + (-2\xi \sigma_+ + 2\eta) + (2\xi^2 + 2\eta^2) \frac{\xi \sigma_+ - \eta}{p} \\ & + \frac{2\xi + 2\eta \sigma_+}{\sqrt{\bar{p}\varphi(\xi^2 + \eta^2 - \bar{p})}} \left\{ \xi^2 + \eta^2 - \bar{p} - \frac{p-p_2}{4\bar{p}}(\xi^2 + \eta^2) \right\} = 0. \end{aligned} \quad (36)$$

The first two terms add to be nonnegative since the direction of obliqueness is pointing inward. The next two terms add to be

$$2(\xi \sigma_+ - \eta) \left(\frac{\xi^2 + \eta^2}{p} - 1 \right) \quad (37)$$

which we show to be nonnegative. First $\xi^2 + \eta^2 \geq p$ at (ξ_0, η_0) . Second, the factor $\xi \sigma_+ - \eta$ is positive since $\sigma_+ \geq 0$ from the previous lemma. Now look at the last term in (36). The factor $\xi + \eta \sigma_+$ can be written as

$$\xi + \eta \sigma_+ = \frac{-\eta(\xi^2 + \eta^2 - \bar{p}) + \xi \sqrt{\bar{p}\varphi}}{\xi(-\eta) + \sqrt{\bar{p}\varphi}(\xi^2 + \eta^2 - \bar{p})}. \quad (38)$$

Using the assumption that $\xi^2 + \eta^2 \geq p$, we see that it is positive. The other factor can be manipulated as

$$\begin{aligned} & \xi^2 + \eta^2 - \bar{p} - \frac{p-p_2}{4\bar{p}}(\xi^2 + \eta^2) \\ & = (\xi^2 + \eta^2) \left\{ 1 - \frac{p-p_2}{4\bar{p}} \right\} - \bar{p} \\ & \geq p \left\{ 1 - \frac{p-p_2}{4\bar{p}} \right\} - \bar{p} \\ & = \frac{p_2(p-p_2)}{2(p+p_2)} > 0. \end{aligned} \quad (39)$$

Thus there is contradiction. So the proof is complete. QED

Lemma 4.5 (*Interior uniform ellipticity*) *The ellipticity holds in the interior*

$$p - \xi^2 - \eta^2 > \epsilon_\delta, \quad \text{in } \Omega_\delta \quad (40)$$

where $\Omega_\delta = \Omega \cap B_r$ where B_r is a ball centered at $(0, 0)$ with radius $r = \sqrt{p_1} - \delta$ for small $\delta > 0$ and $\epsilon_\delta > 0$ is independent of ϵ_e or ϵ_D .

Proof. See Zheng [30], pp. 1860-2.

Lemma 4.6 (*Real property*) *The $\xi^2 + \eta^2 - \bar{p}$ is bounded below by $\frac{3}{4}\epsilon_D^2$ along Σ^+ , provided that*

$$0 < p_1 - p_m \leq c_0, \quad (41)$$

and

$$|\nabla p| \leq c_1 |p_1 - p_m| \quad (42)$$

for some positive constants c_0, c_1 , independent of ϵ_e, ϵ_D , or φ .

Proof. We can calculate the term's value at $\xi = \epsilon_D$, which yields $\frac{7}{8}\epsilon_D^2$; and its value on σ , which yields $(p_1 - p_2)/2$ which is larger than $\frac{7}{8}\epsilon_D^2$. So the term must be increasing at some point along Σ^+ . It therefore may be increasing all the time. We prove that it will not dip below $\frac{6}{8}\epsilon_D^2$, which is sufficient for us. Suppose it reached $\frac{6}{8}\epsilon_D^2$ at the first point $(\xi_1, \eta_1) \in \Sigma^+$. Then the derivative along Σ^+ of the term would be nonpositive at the point. We show that is impossible. Here's the outline. We see

$$\frac{d}{d_{\Sigma}\xi}(\xi^2 + \eta^2)/2 = \xi + \eta\sigma_+ = \frac{-\eta(\xi^2 + \eta^2 - \bar{p}) + \xi\sqrt{\bar{p}\varphi}}{\xi(-\eta) + \sqrt{\bar{p}\varphi}(\xi^2 + \eta^2 - \bar{p})}. \quad (43)$$

We have manipulated that term before, see (38). At (ξ_1, η_1) , we estimate

$$\frac{d}{d_{\Sigma}\xi}(\xi^2 + \eta^2)/2 \approx \min \left\{ \frac{\sqrt{\bar{p}\varphi}}{-\eta_1}, \xi_1 \right\} \geq O^*(\epsilon_D). \quad (44)$$

Examining the oblique derivative condition, we obtain approximately

$$-\frac{[p]}{4} \frac{\dot{p}}{\sqrt{\bar{p}\varphi}} + \nabla p \cdot \mu = 0 \quad (45)$$

where μ is some bounded direction. Once we have (42), then

$$\dot{p} \approx (p_1 - p_m)\sqrt{\varphi} \approx (p_1 - p_m)\epsilon_D. \quad (46)$$

We let $p_1 - p_m$ be small independent of ϵ_e or ϵ_D . This would contradict the dipping of $\xi^2 + \eta^2 - \bar{p}$ at (ξ_1, η_1) . This completes the proof. QED

Thus the modifier φ can be taken away! This is a crucial step, not achieved in previous papers.

We use the now standard norm

$$\|v\|_{H_{1+\alpha}^{(-\gamma)}} = \sup \{ d_B^{1+\alpha-\gamma} (|v|_0 + |\nabla v|_0 + [\nabla v]_{\alpha}) |_{\Omega_{d_B}} \} \quad (47)$$

where d_B stands for distance to the corners B and its counter part A , while Ω_{d_B} is $\Omega \cap B_{d_B}^c(B)$ where $B_{d_B}^c(B)$ is the complement of the ball centered at B with radius d_B .

Lemma 4.7 (*Smallness of slope*) *The slope of the free shock boundary Σ has a bound*

$$|\sigma_{\pm}| \leq c|p_1 - p_m| \quad (48)$$

provided that

$$\|p - p_1\|_{H_{1+\alpha_{\Sigma}}^{(-\gamma)}} \leq c|p_1 - p_m| \quad (49)$$

for some constant c , independent of ϵ_e, ϵ_D , or φ . Assume $\alpha_{\Sigma} < \gamma$.

Proof. We have

$$\eta'(\xi) = \eta'(\epsilon_D) + \int_{\epsilon_D}^{\xi} \eta''(\xi) d\xi \leq \eta'(\epsilon_D) + c \int_{\epsilon_D}^{\xi} |\dot{p}(\xi)| d\xi. \quad (50)$$

We have

$$|\dot{p}| \leq c|p_1 - p_m| d_B^{-(1+\alpha_{\Sigma}-\gamma)}. \quad (51)$$

Since $\alpha_{\Sigma} < \gamma$, we see that $|\dot{p}|$ is integrable. This completes the proof. QED

Now estimate (49) will start to come from linear theory. We comment that we have two obvious corners D and B . The corner B will be bona fide, while corner D is subject to our choice. With the symmetric circular arc, we find that the oblique derivative direction vector at both points point from the exterior of Ω to the exterior. We then use Lieberman's optimal regularity result [20], Theorem 4, to start the regularity estimate with $\gamma \geq 1$. Thus p will be in $H_{1+\alpha_{\Sigma}}(\bar{\Sigma})$ for fixed $\epsilon_e > 0$, and estimate (42) will follow.

4.2 Solution to the core problem

Linear with fixed boundary: See Lieberman [19][20], Guan and Sawyer [14], and Winzell [28], or Canic et al [6].

Nonlinear with fixed boundary: See Popivanov and Kutev [24], or Canic et al [6].

Free boundary: Use Schauder fixed point theorem (Corollary 11.2 of [13]), see Canic, Keyfitz and Kim [6].

4.3 Final limits: Solution to the free boundary problem

The limits are easy to take due to the uniform estimates we have obtained.

5 Further observations

If the angle θ_1 is not close to π , then the solution can be different. In terms of the parameter p_m , we see that p_m can be as small as p_2 . When $p_m = p_2$, the free boundary shock Σ may degenerate partially to a portion of the sonic circle of the state p_2 . See Figure 8. This situation resembles Configuration H in Li et al [17]. We plan to give a proof of this in the near future.

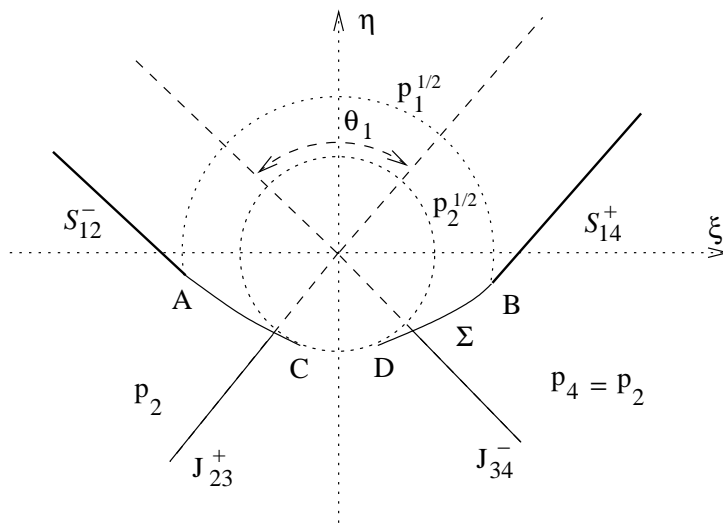


Figure 8. Another type of solution.

Acknowledgment: Discussions with L. C. Evans, G. Lieberman, Jiequan Li, and Tong Zhang are very helpful.

References

- [1] Agarwal, R. K., and Halt, D. W.: A modified CUSP scheme in wave/particle split form for unstructured grid Euler flows, *Frontiers of Computational Fluid Dynamics* (D. A. Caughey and M. M. Hafez, eds.), 1994.
- [2] Bitsadze, A.V., *Boundary value problems for second order elliptic equations*. Translated from the Russian by Scripta Technica, ltd. Translation edited by M. J. Laird.
- [3] Borrelli, R. L., The singular, second order oblique derivative problem, *J. Math. Mech.*, 16(1966), 51–81.
- [4] Borrelli, R. L., The I-Fredholm property for oblique derivative problems. *Math. Z.* 101(1967), 103–122.

- [5] Canic, S., Keyfitz, B., Lieberman, G.: A proof of existence of perturbed steady transonic shocks via a free boundary problem, *Commun Pure Appl Math.*, **53** (2000), 484–511.
- [6] Canic, S., Keyfitz, B., Kim, E.: Free boundary problems for the unsteady transonic small disturbance equation: Transonic regular reflection, *Methods and Applications of Analysis*, 2000
- [7] Canic, S., Keyfitz, B., Kim, E.: A free boundary problem for quasilinear degenerate elliptic equation: Regular reflection of weak shocks, *Commun Pure Appl Math.*, 2002
- [8] Chen, Gui-Qiang, and Feldman, M.: Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, preprint, 2002.
- [9] Chen, Gui-Qiang, and Feldman, M.: Steady transonic shock and free boundary problems in infinite cylinders for the Euler equations, preprint (2003).
- [10] Chen, Shuxing: Linear approximation of shock reflection at a wedge with large angle, *Commun. in Partial Differential Equations*, 21(1996), 1103–1118.
- [11] Dai, Zihuan; and Zhang, Tong: Existence of a global smooth solution for a degenerate Goursat problem of gas dynamics, *Arch. Ration. Mech. Anal.* 155(2000), 277–298.
- [12] Egorov, Ju. V., and Kondrat’ev, V. A., The oblique derivative problem, *mathematics of the ussr-sbornik*, 7(1969), 139–169.
- [13] Gilbarg, D., and Trudinger, N., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, second ed, 1983.
- [14] Guan, Pengfei; Sawyer, E., Regularity estimates for the oblique derivative problem on non-smooth domains (I), *Chin. Ann. of Math.* 16B: 3(1995), 299–324.
- [15] Hörmander, L., Pseudo-differential operators and non-elliptic boundary problems, *Ann. of Math.*, 83(1966), 129–209.
- [16] Kim, E. H., and Song, K., Classical solutions for the pressure-gradient equation in the non-smooth and nonconvex domain, preprint, March, 2003.
- [17] Li, Jiequan, Zhang Tong, Yang Shuli: *The two-dimensional Riemann problem in gas dynamics*, Pitman monographs and surveys in pure and applied mathematics 98. Addison Wesley Longman limited, 1998.
- [18] Kerimov, T. M.: The classical solution of the mixed problem for second order elliptic equations (Russian), *Uspehi Mat. Nauk*, 32(1977), no.6(198), 255–256.
- [19] Lieberman, G.M., Mixed boundary value problems for elliptic and parabolic differential equations of second order, *Journal of Mathematical Analysis and Applications*, 113(1986), 422–440.
- [20] Lieberman, G.M., Optimal Hölder regularity for mixed boundary value problems, *Journal of Mathematical Analysis and Applications*, 143(1989), 572–586.

- [21] Maugeri, A., Palagachev, D. K., and Vitanza, C., A singular boundary value problem for uniformly elliptic operators, *Journal of Mathematical Analysis and Applications*, 263(2001), 33–48. (Note: contact of infinite order, Sobolev-type regularity.)
- [22] Miranda, C., *Partial Differential Equations of Elliptic Type*, 2nd revised edition, Springer, 1970.
- [23] Paneah, B. P., *The Oblique Derivative Problem The Poincaré Problem*, Wiley-VCH, 2000.
- [24] Popivanov, P., Kutev, N., The tangential oblique derivative problem for nonlinear elliptic equations, *Comm. in Partial Differential Equations*, 14(1989), 413–428.
- [25] Popivanov, P., Kutev, N., Viscosity solutions to the degenerate oblique derivative problem for fully nonlinear elliptic equations, *C. R. Acad. Sci. Paris, Ser. I* 334(2002), pp. 661–666.
- [26] Popivanov, P. R. and Palagachev, D. K., *The Degenerate Oblique Derivative Problem for Elliptic and Parabolic Equations*, Academic Verlag, VCH Publishers, Inc. Berlin-New York 1997.
- [27] Song, Kyungwoo, A pressure-gradient system on non-smooth domains, *Comm. in Partial Differential Equations*, to appear, 2003.
- [28] Winzell, B., A boundary value problem with an oblique derivative, *Comm. in Partial Differential Equations*, 6(1981), 305–328.
- [29] Zaremba, S., Sur un problème mixte relatif a l'équation de Laplace, *Bull. Int. Acad. Cracovie, Classe Math. Nat., Ser. A*, (1910), 313–344.
- [30] Zheng, Yuxi: Existence of solutions to the transonic pressure-gradient equations of the compressible Euler equations in elliptic regions, *Communications in Partial Differential Equations*, 22(1997), 1849–1868.
- [31] Zheng, Yuxi: *Systems of Conservation Laws: Two-Dimensional Riemann Problems*, 38 PNLDE, Birkhäuser, Boston, 2001