EXISTENCE OF SOLUTIONS TO THE TRANSONIC PRESSURE-GRADIENT EQUATIONS OF THE COMPRESSIBLE EULER EQUATIONS IN ELLIPTIC REGIONS

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Abstract. We establish the existence of a smooth solution in its elliptic region in the self-similar plane to the pressure-gradient system arisen from the wave-particle splitting of the two-dimensional compressible Euler system of equations. The pressure-gradient system takes the form $\rho u_t + p_x = 0$, $\rho v_t + p_y = 0$, $\rho E_t + (up)_x + (vp)_y = 0$. Here (u, v) is the velocity, ρ is the density which is independent of time resulted from the splitting procedure, p is the pressure, and $E = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1}\frac{p}{\rho}$ is the energy. The natural (parabolically degenerate) boundary value is used.

I. Introduction

There are simple and interesting systems of conservation laws involved in flux-splitting schemes for the two-dimensional (2-D) compressible Euler equations. Separating the pressure from the inertia in the flux of the Euler equations

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uE + up \end{bmatrix}_x + \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vE + vp \end{bmatrix}_y = 0, \tag{1.1}$$

where

$$E = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1}\frac{p}{\rho},$$

we obtain two systems of equations

$$(1.2)\qquad \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 \\ \rho uv \\ \rho uE \end{bmatrix}_x + \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 \\ \rho vE \end{bmatrix}_y = 0$$

and

$$(1.3)\qquad \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}_t + \begin{bmatrix} 0 \\ p \\ 0 \\ up \end{bmatrix}_x + \begin{bmatrix} 0 \\ 0 \\ p \\ vp \end{bmatrix}_y = 0\,.$$

Agarwal and Halt [1] have recently used this splitting (1.2-3) to form a novel scheme in numerical computations for airfoil flows and observed a consistent improvement over other schemes (Roe, AUSM, CUSP, and Van Leer). Systems (1.2) and (1.3) may be called the *transport* and the *pressure-gradient systems* respectively.

We are more interested here in system (1.3), which resembles (1.1) more than (1.2) does in their essential nonlinear structures. We simplify (1.3) slightly. From the first equation of system (1.3) we obtain

$$\rho_t = 0.$$

Thus ρ is independent of time. For simplicity, we will assume

$$\rho = 1.$$

Then system (1.3) can be written as

$$(1.4)\qquad \begin{cases} u_t + p_x = 0 \\ v_t + p_y = 0 \\ E_t + (up)_x + (vp)_y = 0 \end{cases}$$

where $E = (u^2+v^2)/2 + p/(\gamma-1)$. For smooth solutions or in regions where a solution is smooth, system (1.4) can be simplified to be

$$(1.5)\qquad \begin{cases} u_t + p_x = 0 \\ v_t + p_y = 0 \\ \frac{1}{\gamma-1}p_t + pu_x + pv_y = 0. \end{cases}$$

Through the transformation

$$\begin{cases} p = (\gamma - 1)P \\ t = \frac{1}{\gamma - 1} T, \end{cases}$$

system (1.5) can be rewritten to be

$$\begin{cases} u_T + P_x = 0 \\ v_T + P_y = 0 \\ P_T + P u_x + P v_y = 0. \end{cases} \tag{1.6}$$

From system (1.6) we can find that

$$\boxed{\left(\frac{P_T}{P}\right)_T = P_{xx} + P_{yy}} \tag{1.7}$$

which we think is a very interesting, and one of the simplest, 2-D quasilinear wave equation.

There is less progress in the study of Cauchy problems for systems (1.3-7) than for system (1.2). See [2-4, 7-8, 10, 19, 27] for progress in system (1.2). In particular, we note from the interesting work of E, Rykov, and Sinai ([10]) and references therein that system (1.2) is a valid physical model in adhesion particle dynamics. For systems (1.3), (1.4), or (1.7), however, even Riemann-type problems are open. Peng Zhang, Jiequan Li, and Tong Zhang [31] are currently working on a set of conjectures for the solutions to a four-constant Riemann problem for these systems. They use, as usual, the self-similar coordinates

$$\xi = \frac{x}{T}, \quad \eta = \frac{y}{T}$$

to reduce the Riemann problem by one dimension. In these coordinates, however, all the systems (1.3), (1.4), and (1.7), or even their linearized versions are of mixed type. One major difficulty in proving their conjectures is this change of type combined with the nonlinearity of the systems. Our interest here is to establish the existence of solutions in the elliptic region.

Since the existence of solutions in the elliptic region is a big open problem for the original 2-D Euler equation (1.1) (but see the more recent papers of Morawetz [21-24] and Kan and Chen [16] for existence results for steady and irrotational flows), we are also interested in using (1.3) or (1.7) as a simple time-dependent model for establishing mathematical tools to handle general transonic flows.

Equation (1.7) in the self-similar coordinates (ξ, η) takes the form

(1.8)
$$\boxed{(P-\xi^2)P_{\xi\xi} - 2\xi\eta P_{\xi\eta} + (P-\eta^2)P_{\eta\eta} + \frac{(\xi P_\xi + \eta P_\eta)^2}{P} - 2(\xi P_\xi + \eta P_\eta) = 0.}$$

The eigenvalues of the coefficient matrix of the second order terms of (1.8) can be found to be P and $P - \xi^2 - \eta^2$. We prove in this paper the existence of a weak solution for equation (1.8) in any open, bounded and convex region $\Omega \subset \mathbb{R}^2$ with smooth boundary and the degenerate boundary datum

$$\boxed{P|_{\partial\Omega} = \xi^2 + \eta^2} \tag{1.9}$$

provided that the boundary of Ω does not contain the origin $(0, 0)$.

In this area of 2-D Riemann problems for the compressible Euler equations, there have been many models including the 2-D complex Burgers of Lax [18], the unsteady transonic small disturbance system [15, 25, 28](Hunter, Morawetz, Tabak & Rosales), the potential flow equation [20] (Majda), and a 2×2 system of Tan and Zhang [29]. In particular, Čanić and Keyfitz [5,6] have obtained the existence with regularity of weak solutions for the unsteady transonic small disturbance system with partial degenerate boundary conditions. See Choi and others [9] for the existence and regularity results of solutions to a related equation ($u^a u_{xx} + u^b u_{yy} + p(x, y) = 0$ with $u = 0$ on the boundary, where $a > b \geq 0$ are constants).

Our approach to existence of weak solutions is different from that of [5,6], but close to that of [9]; both our approach and that of [9] seem to have more difficulty in handling nonsmooth boundaries than that of [5,6]. In particular, our result here gives less accurate regularity estimates near boundaries than those of [5,6]. On the other hand, the approach taken here seems more direct.

There is no general theory on boundary value problems for degenerate nonlinear elliptic equations such as (1.8–9). For linear equations, we are aware of interesting work on Tricomi's [30] and Keldysh's equations [17] and Fichera's theory on Keldysh-type of equations [12,13]. Although it is not clear whether Fichera's linear theory can apply to nonlinear cases, we do find trivially that the degenerate boundary value problem (1.8–9) satisfies Fichera's condition for well-posedness when it is linearized around constant solutions on circular domains containing the origin. See Oleĭnik and Radkevič [26] or Čanić and Keyfitz [5] for Fichera's condition, besides Fichera's original work. On circular domains not containing the origin, however, we find that Fichera's condition for well-posedness is violated on some part of the boundary when problem (1.8–9) is linearized around constant solutions.

Although system (1.3) is used in conjunction with system (1.2) in the flux-splitting scheme, we find that (1.3) (and its simplified systems (1.4-7)) is formally asymptotically valid when the velocity (u, v) is small and the gas constant γ is large compared to the density and pressure gradient of the flow. In fact, we find $\rho_t = 0$ from the first equation of 2-D Euler system (1.1) when (u, v) is small. Thus we assume similarly $\rho = 1$ for brevity. Then we obtain the first two equations of (1.4) from the second and third equations of system (1.1) because the quadratic terms such as $(\rho u^2)_x$ are smaller than the linear terms such as $(\rho u)_t$ or the pressure-gradient terms such as p_x which are assumed to be large. Furthermore we find

$$\rho E = \frac{1}{2}\rho(u^2 + v^2) + \frac{1}{\gamma - 1}p \ll p$$

when (u, v) are small and γ is large. So we can drop the terms containing E in the flux of (1.1) to yield system (1.4). Thus, system (1.3) may have its own physical region of validity.

II. The motivational Riemann problem

We motivate our problem (1.8) (1.9) more specifically in this section.

We propose a four-constant initial value problem for system (1.6) in such a way as to yield, conceivably, a continuous solution for $t > 0$. Consider the data

$$(u, v, P)\big|_{T=0} = (u_i, v_i, P_i), \text{ for } (x, y) \text{ in the } i^{\text{th}} \text{ quadrant}, \tag{2.1}$$

for $i = 1, 2, 3, 4$, where $\{(u_i, v_i, P_i)\}_{i=1}^4$ are constants. In order for the neighboring two states (u_1, v_1, P_1) and (u_2, v_2, P_2) to be connected continuously, we find that there must hold

$$\begin{cases} v_2 = v_1 \\ u_2 - u_1 = \pm 2\left(\sqrt{P_2} - \sqrt{P_1}\right). \end{cases} \tag{2.2}$$

For our motivational purpose, we choose $P_2 < P_1$ and the plus sign in (2.2) which then gives the forward rarefaction wave in one space dimension. We may fix the state (u_1, v_1, P_1) and treat P_2 as a free variable, and (u_2, v_2) follows from (2.2). The rarefaction wave is shown in Fig. 2.1 between the first and second quadrants. The value of (u, P) in the rarefaction wave (the region of parallel lines) is given by

$$\begin{cases} P = \xi^2 \\ u = 2(\xi - \sqrt{P_1}) + u_1, \quad \sqrt{P_2} < \xi < \sqrt{P_1}. \end{cases} \tag{2.3}$$

Similarly the one-dimensional rarefaction waves between states 1 and 4, 2 and 3, and 3 and 4 can be found to satisfy

$$\begin{cases} u_4 = u_1 \\ v_4 - v_1 = 2(\sqrt{P_4} - \sqrt{P_1}) \end{cases} \qquad P_4 < P_1 \tag{2.4}$$

$$\begin{cases} u_3 = u_2 \ \left(= u_1 + 2(\sqrt{P_2} - \sqrt{P_1})\right) \\ v_3 - v_1 = 2(\sqrt{P_3} - \sqrt{P_2}) \end{cases} \qquad P_3 < P_2 \tag{2.5}$$

$$(2.6)\qquad \begin{cases} v_3 = v_4 \ \left(= v_1 + 2(\sqrt{P_4} - \sqrt{P_1})\right) \\ u_3 - u_4 = 2(\sqrt{P_3} - \sqrt{P_4}) \end{cases} \qquad P_3 < P_4.$$

For (2.5) (2.6) to be consistent, we need the necessary compatibility condition

$$(2.7)\qquad \sqrt{P_1} - \sqrt{P_2} = \sqrt{P_4} - \sqrt{P_3}$$

which is also sufficient.

The waves between states 1 and 2, and 1 and 4 coming from infinity begin to interact at the point $(\xi, \eta) = (\sqrt{P_1}, \sqrt{P_1})$. The two characteristics from the point, and possibly also part of the sonic circle of the state 3, form a loop which separates the (ξ, η)-plane into two regions. The exterior region consists of the four rarefaction waves and constant states. The interior region is where the solution needs to be constructed. We speculate that the interior region consists also of two regions, one is elliptic (the region inside the dashed line in Fig. 2.1.) and the other is hyperbolic where the four rarefaction waves interact. Characteristics in the hyperbolic region and on the parabolic curve may look like what is shown in Fig. 2.1. The parabolic curve may be expected to be convex and smooth. Some part of the parabolic curve may be circular. On all of the curve there holds the relation

$$(2.8)\qquad P = \xi^2 + \eta^2.$$

III. The result

We consider the problem: Find a weak or smooth solution $u(x, y)$ to the problem

$$(3.1)\qquad \begin{cases} (u - x^2)\dfrac{\partial^2 u}{\partial x^2} - 2xy\dfrac{\partial^2 u}{\partial x \partial y} + (u - y^2)\dfrac{\partial^2 u}{\partial y^2} + \\ \qquad\qquad + \dfrac{1}{u}\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right)^2 - 2\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = x^2 + y^2, \end{cases}$$

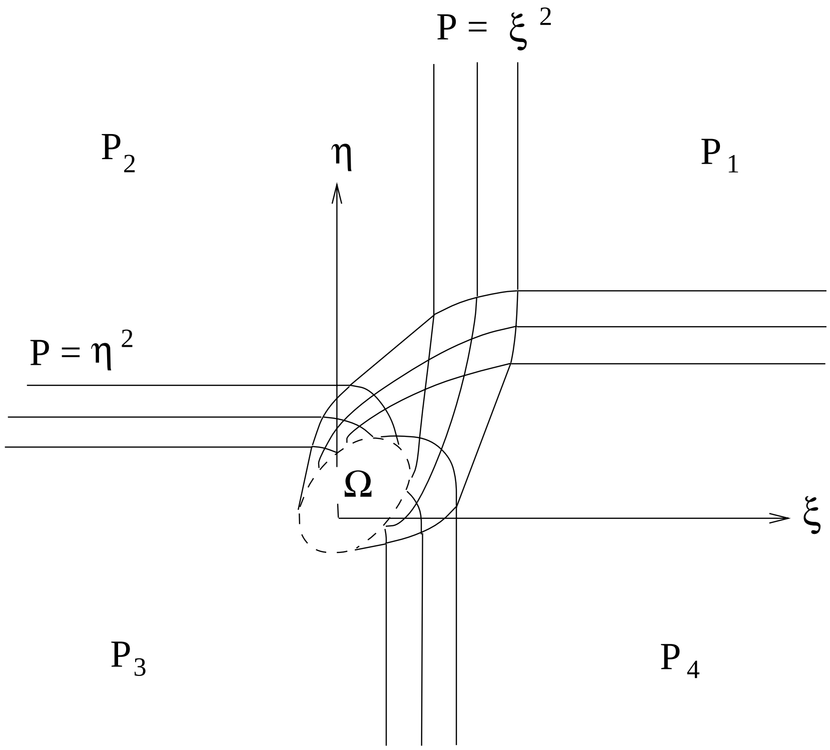

FIG. 2.1: An elliptic region Ω in the solution of a Riemann problem.

where the region $\Omega \subset \mathbb{R}^2$ is open, bounded, and convex with boundary $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0,1)$.

We assume that the origin $(0,0)$ does not lie on the boundary of Ω.

Our result is

Theorem. *There exists a positive weak solution* $u \in H^1_{loc}(\Omega)$ *to the problem (3.1) with* $u \in C^{0,\alpha}_{loc}(\Omega)$. *It takes on the boundary value in the sense* $[u - (x^2+y^2)]^{3/2} \in H^1_0(\Omega)$. *Furthermore, it has*

(i) *maximum principle:* $\min_{\partial\Omega}(x^2+y^2) \le u(x,y) \le \max_{\partial\Omega}(x^2+y^2)$

(ii) *interior ellipticity:* $u(x,y) - (x^2+y^2) > 0$ *in* Ω.

Remark. $H^1_{loc}(\Omega)$ denotes the space of all functions $u \in H^1(\Omega')$ for any $\Omega' \subset\subset \Omega$, i.e. $\Omega' \subset \Omega$ and the closure $\overline{\Omega'} \subset \Omega$. $C^{0,\alpha}_{loc}(\Omega)$ means the same.

Proof. **1.** We introduce the function

$$K(x,y,z) = \begin{cases} z, & \text{if } z \ge x^2+y^2, \\ x^2+y^2, & \text{if } z < x^2+y^2. \end{cases}$$

This function is Lipschitz continuous in $\mathbb{R}^3$. Now consider *the screened problem*

$$
(3.2) \qquad \begin{cases} (K(x,y,u) - x^2 + \varepsilon)u_{xx} - 2xyu_{xy} + (K(x,y,u) - y^2 + \varepsilon)u_{yy} \\ \qquad\qquad + \frac{1}{N(u)}\left(xu_x + yu_y\right)^2 - 2\left(xu_x + yu_y\right) = 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} = x^2 + y^2 \end{cases}
$$

where $\varepsilon > 0$ is a parameter, $N(v) \in C^3(\mathbb{R})$ is bounded from both above and below with a positive lower bound, and $N(v) = v$ in the interval $\left[\min_{\partial\Omega}(x^2+y^2), \max_{\partial\Omega}(x^2+y^2)\right]$.

We find easily that the equation in (3.2) is uniformly elliptic in Ω, since the two eigenvalues of the matrix

$$
A = \begin{pmatrix} K - x^2 + \varepsilon & -xy \\ -xy & K - y^2 + \varepsilon \end{pmatrix}
$$

are $\Lambda = K(x,y,u) + \varepsilon$, and $\lambda = K(x,y,u) + \varepsilon - (x^2+y^2) \geq \varepsilon$. Then the existence of a $C^{2,\alpha}(\overline{\Omega})$ solution $u^{K,\varepsilon,N}(x,y)$ of (3.2) follows from Theorem 15.12 of Gilbarg and Trudinger [14] in the special case of two space dimensions in which the Hölder continuity of K is sufficient.

2. The solutions $u^{K,\varepsilon,N}$ satisfy the maximum principle

$$
\min_{\partial\Omega}(x^2+y^2) \leq u^{K,\varepsilon,N}(x,y) \leq \max_{\partial\Omega}(x^2+y^2) \quad \text{in } \Omega,
$$

see Gilbarg-Trudinger [14]. Therefore the $N(u)$ regularization degenerates to u itself. So the solutions $u^{K,\varepsilon,N}$ are independent of N. We therefore drop the superscript N. We thus have $C^{2,\alpha}(\overline{\Omega})$ solutions $u^{K,\varepsilon}$ to the problem

$$
(3.3) \qquad \begin{cases} (K(x,y,u) - x^2 + \varepsilon)u_{xx} - 2xyu_{xy} + (K(x,y,u) - y^2 + \varepsilon)u_{yy} \\ \qquad\qquad + \frac{1}{u}\left(xu_x + yu_y\right)^2 - 2(xu_x + yu_y) = 0, \\ u|_{\partial\Omega} = x^2 + y^2. \end{cases}
$$

We next show that the $K(x,y,u)$ screening in fact degenerates to u also for the solutions $u^{K,\varepsilon}(x,y)$. Let

$$
F(x,y) = u^{K,\varepsilon}(x,y) - (x^2+y^2).
$$

We have $F|_{\partial\Omega} = 0$. We claim that $F > 0$ in Ω. Otherwise there exists a point $(x_0, y_0) \in \Omega$, which may depend on (K, ε), such that $F(x_0, y_0) \leq 0$, $F_x(x_0, y_0) = F_y(x_0, y_0) = 0$, and

$$(K(x,y,u^{K,\varepsilon}(x,y)) - x^2 + \varepsilon)F_{xx} - 2xyF_{xy} + (K(x,y,u^{K,\varepsilon}(x,y)) - y^2 + \varepsilon)F_{yy} \geq 0$$

at (x_0, y_0). Rewriting those conditions in terms of u and using equation (3.3), we find

$$u^{K,\varepsilon}(x_0, y_0) \leq x_0^2 + y_0^2$$
$$\frac{\partial u^{K,\varepsilon}}{\partial x}(x_0, y_0) = 2x_0, \qquad \frac{\partial u^{K,\varepsilon}}{\partial y}(x_0, y_0) = 2y_0,$$

and

$$\begin{aligned} 2(K(x,y,&u^{K,\varepsilon}(x,y)) - x^2 + \varepsilon) + 2(K(x,y,u^{K,\varepsilon}(x,y)) - y^2 + \varepsilon) \\ &+ \frac{(2x^2+2y^2)^2}{u^{K,\varepsilon}} - 2(2x^2+2y^2) \leq 0 \quad \text{at } (x_0, y_0). \end{aligned}$$

But the last inequality is false because

$$\begin{aligned} &K(x,y,u^{K,\varepsilon}(x,y)) - x^2 + \varepsilon \geq \varepsilon \\ &K(x,y,u^{K,\varepsilon}(x,y)) - y^2 + \varepsilon \geq \varepsilon \\ &\frac{(2x_0^2+2y_0^2)^2}{u^{K,\varepsilon}(x_0,y_0)} - 2(2x_0^2+2y_0^2) \geq \frac{(2x_0^2+2y_0^2)^2}{x_0^2+y_0^2} - 2(2x_0^2+2y_0^2) \geq 0. \end{aligned}$$

Hence we have proved that

$$u^{K,\varepsilon}(x,y) > x^2 + y^2 \quad \text{in } \Omega.$$

Therefore our solutions $u^{K,\varepsilon}$ do not even depend on K. So we drop the K dependence and we have existence of $C^{2,\alpha}(\overline{\Omega})$ solutions $u^\varepsilon(x,y)$ to the problem

$$(3.4) \qquad \begin{cases} (u - x^2 + \varepsilon)u_{xx} - 2xyu_{xy} + (u - y^2 + \varepsilon)u_{yy} + \\ \qquad\qquad + \dfrac{1}{u}\left(xu_x + yu_y\right)^2 - 2\left(xu_x + yu_y\right) = 0 \\ u|_{\partial\Omega} = x^2 + y^2. \end{cases}$$

3. We now establish one of the two major estimates on u^ε independently of $\varepsilon > 0$. We introduce the function

$$\varphi^\varepsilon(x, y) = u^\varepsilon - (x^2 + y^2).$$

We find that φ^ε satisfies the equation

$$(3.5)\quad \begin{cases} (\varepsilon + \varphi + y^2)\varphi_{xx} - 2xy\varphi_{xy} + (\varepsilon + \varphi + x^2)\varphi_{yy} + \frac{(x\varphi_x + y\varphi_y - 2\varphi)^2}{\varphi + x^2 + y^2} \\ \qquad\qquad +2\,(x\varphi_x + y\varphi_y) + 2x^2 + 2y^2 + 4\varepsilon = 0, \\ \varphi|_{\partial\Omega} = 0, \end{cases}$$

where we have dropped the superscript ε for simplicity. We know that $0 < \varphi^\varepsilon \le \max_{\partial\Omega}(x^2 + y^2)$ in Ω. We show that

$$(3.6)\qquad \iint_\Omega \varphi^\varepsilon |D\varphi^\varepsilon|^2 dxdy \le C \quad \text{independent of } 1 \ge \varepsilon > 0.$$

In fact, we multiply equation (3.5) with φ^ε to find

$$\begin{aligned} &[(\varphi + y^2 + \varepsilon)\varphi\varphi_x - xy\varphi\varphi_y]_x + [-xy\varphi\varphi_x + (\varphi + x^2 + \varepsilon)\varphi\varphi_y]_y \\ &\quad - (2\varphi + y^2 + \varepsilon)\varphi_x^2 + 2xy\varphi_x\varphi_y - (2\varphi + x^2 + \varepsilon)\varphi_y^2 + \frac{\varphi(x\varphi_x + y\varphi_y)^2}{\varphi + x^2 + y^2} \\ &\qquad + 3\varphi(x\varphi_x + y\varphi_y) - \frac{4\varphi^2}{\varphi + x^2 + y^2}(x\varphi_x + y\varphi_y) + \frac{4\varphi^3}{\varphi + x^2 + y^2} \\ &\qquad\quad + 2\varphi(x^2 + y^2) + 4\varphi\varepsilon = 0. \end{aligned}$$

Integrating over Ω and using the zero boundary condition of φ, we obtain

$$\begin{aligned} &\iint_\Omega \Big[(2\varphi + y^2 + \varepsilon)\varphi_x^2 - 2xy\varphi_x\varphi_y + \\ &\qquad\qquad (2\varphi + x^2 + \varepsilon)\varphi_y^2 - \frac{\varphi(x\varphi_x + y\varphi_y)^2}{\varphi + x^2 + y^2}\Big] dxdy \\ &= \iint \Big[\Big(3\varphi - \frac{4\varphi^2}{\varphi + x^2 + y^2}\Big)(x\varphi_x + y\varphi_y) + \\ &\qquad\qquad \frac{4\varphi^3}{\varphi + x^2 + y^2} + 2\varphi(x^2 + y^2) + 4\varphi\varepsilon\Big] dxdy. \end{aligned}$$

We further simplify the integral on the left-hand side in the above equation to obtain

$$\iint_\Omega \frac{2\varphi + x^2 + y^2}{\varphi + x^2 + y^2}[(\varphi + y^2)\varphi_x^2 - 2xy\varphi_x\varphi_y + (\varphi + x^2)\varphi_y^2]dxdy + \varepsilon \iint_\Omega (\varphi_x^2 + \varphi_y^2)dxdy$$
$$= \iint_\Omega \left[\left(3\varphi - \frac{4\varphi^2}{\varphi + x^2 + y^2}\right)(x\varphi_x + y\varphi_y) + \frac{4\varphi^3}{\varphi + x^2 + y^2} + 2\varphi(x^2 + y^2) + 4\varphi\varepsilon\right] dxdy.$$

Using the fact $\varphi > 0$ in Ω and is bounded from above by $\max_{\partial\Omega}(x^2 + y^2)$, we find

$$\iint_\Omega \varphi(\varphi_x^2 + \varphi_y^2) \le C \iint_\Omega \varphi(|\varphi_x| + |\varphi_y|)\, dxdy + C$$

where C is a constant depending on Ω, but independent of $\varepsilon \in (0, 1]$.

A weighted Cauchy-Schwarz inequality on the right-hand side yields

$$\iint_\Omega \varphi(\varphi_x^2 + \varphi_y^2)\, dxdy \le \frac{1}{2} \iint_\Omega \varphi(\varphi_x^2 + \varphi_y^2)\, dxdy + C$$

which further yields

$$\iint_\Omega \varphi^\varepsilon(\varphi_x^{\varepsilon 2} + \varphi_y^{\varepsilon 2})\, dxdy \le C(\Omega).$$

So (3.6) is proved.

4. We need our next major estimate to be able to draw convergent subsequences of φ^ε in $H^1_{loc}(\Omega)$ from the above estimate. We establish ellipticity of u^ε uniformly for $\varepsilon \in (0, 1]$ in the interior of Ω. Let $\xi(x, y)$ be any nonnegative function in $C^3(\Omega)$ with zero boundary data $\xi|_{\partial\Omega} = 0$. We claim that there exists a small positive number $\beta > 0$, independent of $\varepsilon \in (0, 1]$ such that

$$\eta^\varepsilon(x, y) \equiv u^\varepsilon(x, y) - (x^2 + y^2) - \beta\xi(x, y) > 0 \quad \text{in } \Omega. \tag{3.7}$$

In fact, we can take $\beta > 0$ so small that

$$\beta\left(2\max_{\partial\Omega}(x^2+y^2)+1\right)\max_{\Omega}|D^2\xi| < \min_{\partial\Omega}(x^2+y^2) \tag{3.8}$$

where $D^2\xi$ represents all second order derivatives, and

$$\beta\max_{\Omega}\left|\frac{2x^2+2y^2+\beta x\xi_x+\beta y\xi_y}{x^2+y^2+\beta\xi}(x\xi_x+y\xi_y-2\xi)\right| < \min_{\partial\Omega}(x^2+y^2). \tag{3.9}$$

We note that $\min_{\partial\Omega}(x^2+y^2) > 0$ since the origin $(0,0)$ does not belong to $\partial\Omega$. We use contradiction method to prove (3.7). Suppose $\eta^\varepsilon(x,y)$ is not positive in Ω. There must be a minimum point in the interior of Ω. At the minimum point,

$$\begin{aligned}
u^\varepsilon &\le x^2+y^2+\beta\xi(x,y)\\
u^\varepsilon_x &= 2x+\beta\xi_x\\
u^\varepsilon_y &= 2y+\beta\xi_y,
\end{aligned}$$

and

$$(u^\varepsilon-x^2+\varepsilon)\eta^\varepsilon_{xx}-2xy\eta^\varepsilon_{xy}+(u^\varepsilon-y^2+\varepsilon)\eta^\varepsilon_{yy}\ge 0.$$

Using equation (3.4) for u^ε, we find

$$\begin{aligned}
&(u^\varepsilon-x^2+\varepsilon)\eta^\varepsilon_{xx}-2xy\eta^\varepsilon_{xy}+(u^\varepsilon-y^2+\varepsilon)\eta^\varepsilon_{yy}\\
&\quad+2(u^\varepsilon-x^2+\varepsilon)+2(u^\varepsilon-y^2+\varepsilon)\\
&\quad+\beta\left[(u^\varepsilon-x^2+\varepsilon)\xi_{xx}-2xy\xi_{xy}+(u^\varepsilon-y^2+\varepsilon)\xi_{yy}\right]\\
&\quad+\frac{(2x^2+2y^2+\beta x\xi_x+\beta y\xi_y)^2}{\eta^\varepsilon+(x^2+y^2)+\beta\xi}\\
&\quad-2(2x^2+2y^2+\beta x\xi_x+\beta y\xi_y)=0.
\end{aligned} \tag{3.10}$$

We observe that in the above equation, the first three terms together is nonnegative; the next two terms together gives

$$\begin{aligned}
2(u^\varepsilon-x^2+\varepsilon)+2(u^\varepsilon-y^2+\varepsilon) &\ge 2(u^\varepsilon-x^2-y^2)+2u^\varepsilon\\
&\ge 2u^\varepsilon \ge 2\min_{\partial\Omega}(x^2+y^2)>0.
\end{aligned}$$

The terms in the bracket can be bounded by

$$
\begin{aligned}
\beta \big| \big[(u^\varepsilon - x^2 + \varepsilon)\xi_{xx} - 2xy\xi_{xy} + (u^\varepsilon - y^2 + \varepsilon)\xi_{yy} \big] \big| \\
&\le \beta \left(2 \max_{\partial\Omega}(x^2+y^2) + 1 \right) \max_{\Omega} |D^2 \xi| \\
&\le \min_{\partial\Omega}(x^2+y^2)
\end{aligned}
$$

where we used (3.8). All the remaining terms in the equation have the estimate

$$
\begin{aligned}
&\frac{(2x^2+2y^2+\beta x\xi_x+\beta y\xi_y)^2}{\eta^\varepsilon+\beta\xi+x^2+y^2} - 2(2x^2+2y^2+\beta x\xi_x+\beta y\xi_y) \\
&\ge \frac{(2x^2+2y^2+\beta x\xi_x+\beta y\xi_y)^2}{x^2+y^2+\beta\xi} - 2(2x^2+2y^2+\beta x\xi_x+\beta y\xi_y) \\
&= \beta \frac{2x^2+2y^2+\beta x\xi_x+\beta y\xi_y}{x^2+y^2+\beta\xi}(x\xi_x+y\xi_y-2\xi)
\end{aligned}
$$

whose absolute value is less than $\min_{\partial\Omega}(x^2+y^2)$ by (3.9). Thus the equation for η^ε (3.10) is violated. Hence $\eta^\varepsilon > 0$ in Ω. Therefore

$$
u^\varepsilon(x,y) - (x^2+y^2) \ge \beta\xi(x,y) > 0 \quad \text{in } \Omega
$$

for a small $\beta > 0$ independent of $\varepsilon > 0$.

5. Hence in any $\Omega' \subset\subset \Omega$, we have

$$
\iint_{\Omega'} |D\varphi^\varepsilon|^2 dxdy \le C.
$$

We therefore have a subsequence of $\{\varphi^\varepsilon\}_{\varepsilon>0}$, still denoted by $\{\varphi^\varepsilon\}_{\varepsilon>0}$ which converges weakly to a function $\varphi \in H^1_{loc}(\Omega)$

$$
\begin{aligned}
\varphi^\varepsilon &\longrightarrow \varphi \quad \text{in } L^2_{loc}(\Omega), \\
\nabla\varphi^\varepsilon &\rightharpoonup \nabla\varphi \quad \text{in } L^2_{loc}(\Omega).
\end{aligned}
$$

Using a theorem found in Evans [11] (see the appendix for more details), we can improve the weak convergence to strong convergence:

$$
\nabla\varphi^\varepsilon \longrightarrow \nabla\varphi \text{ in } L^2_{loc}(\Omega).
$$

Thus φ satisfies the equation in problem (3.1) in the sense of distributions. We have trivially

$$\iint_\Omega \left|D\left((\varphi^\varepsilon)^{3/2}\right)\right|^2 dxdy \le C, \quad (\varphi^\varepsilon)^{3/2}\Big|_{\partial\Omega} = 0.$$

It follows that

$$\iint_\Omega \left|D\left(\varphi^{3/2}\right)\right|^2 dxdy \le C, \text{ and } \varphi^{3/2}\Big|_{\partial\Omega} = 0 \text{ in trace sense.}$$

Therefore

$$\varphi^{3/2} \in H_0^1(\Omega).$$

By Lemma 15.4 of Gilbarg-Trudinger [14] (i.e., the Krylov estimate), we obtain the interior estimate $u \in C^{0,\alpha}_{loc}(\Omega)$.

Remark. We do not obtain higher regularity than $u \in C^{0,\alpha}_{loc}(\Omega)$ despite that the equation in (3.1) is elliptic in the interior, because the quadratic nonlinear term $(xu_x + yu_y)^2$ is only in L^1.

Appendix

We improve the weak convergence in $W^{1,2}_{\text{loc}}(\Omega)$ of the sequence of approximate solutions in Section III to strong convergence.

Consider a sequence of second order quasilinear equations

$$(A.1) \qquad \sum_{i,j=1}^{n} a^\varepsilon_{ij}(x,\varphi)\varphi_{x_i x_j} + B^\varepsilon(x,\varphi,D\varphi) = 0, \quad 0 < \varepsilon \le 1$$

where $x \in \Omega \subset \mathbb{R}^n$ for some open set Ω. We assume that B^ε and $(a^\varepsilon_{ij})^n_{i,j=1}$ are smooth functions of their arguments, including the argument $\varepsilon \in [0,1]$. In addition, we assume

$$(A.2) \qquad \left(a^\varepsilon_{ij}(x,\varphi)\right)^n_{ij=1} \ge \lambda(\varphi)\mathbb{I}$$

$$(A.3) \qquad |B^\varepsilon(x,\varphi,p)| \le C_M(1+|p|^2)$$

for all $0 < \varepsilon \leq 1,\ x \in \Omega,\ |\varphi| \leq M$, and $p \in \mathbb{R}^n$. The smallest eigenvalue $\lambda(\varphi)$ is assumed to satisfy

$$(A.4) \qquad \lambda(\varphi) > 0 \quad \text{if} \quad \varphi > 0.$$

For our application in Sect. III, this condition (A.4) is satisfied since we have $\lambda(\varphi) = \varphi$.

Assume we have for (A.1) a sequence of approximate solutions $\{\varphi^\varepsilon\}_{\varepsilon>0}$ satisfying

$$(A.5) \qquad 0 < \xi(x) < \varphi^\varepsilon(x) \leq M, \quad x \in \Omega$$

$$(A.6) \qquad \int_\Omega \varphi^\varepsilon |D\varphi^\varepsilon|^2 dx \leq C$$

$$(A.7) \qquad \varphi^\varepsilon \to \varphi \text{ in } L^2_{\text{loc}}(\Omega)$$

$$(A.8) \qquad D\varphi^\varepsilon \rightharpoonup D\varphi \text{ in } L^2_{\text{loc}}(\Omega) \text{ weakly}$$

for some constants M and C, a smooth function $\xi(x)$ and a limit $\varphi \in W^{1,2}_{\text{loc}}(\Omega)$. We establish in this appendix the strong convergence

$$(A.9) \qquad D\varphi^\varepsilon \to D\varphi \text{ in } L^2_{\text{loc}}(\Omega),$$

i.e.,

$$D\varphi^\varepsilon \to D\varphi \text{ in } L^2(\Omega'), \quad \forall\, \Omega' \subset\subset \Omega.$$

As in Evans [11, p.40], we use the nonlinear test function method. Since our equations (A.1) may be degenerate near the boundary, we choose to have a cut-off factor in the test function. For any set $\Omega' \subset\subset \Omega$, we pick an arbitrary non-negative function, $\zeta \in C_c^\infty(\Omega)$ such that $\zeta = 1$ in Ω'. For any $v \in L^\infty(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$, we have

$$
\begin{aligned}
(A.10) \qquad & \int_\Omega \sum_{i,j=1}^n \zeta a^\varepsilon_{ij}(x,\varphi^\varepsilon)\varphi^\varepsilon_{x_i} v_{x_j}\,dx + \int_\Omega v \sum_{i,j=1}^n a^\varepsilon_{ij}(x,\varphi^\varepsilon)\varphi^\varepsilon_{x_i}\zeta_{x_j}\,dx \\
& \qquad = \int_\Omega \overline{B}^\varepsilon(x,\varphi^\varepsilon,D\varphi^\varepsilon)\zeta v dx
\end{aligned}
$$

where we have used the notation

$$
(A.11) \qquad B^\varepsilon(x,\varphi^\varepsilon,D\varphi^\varepsilon) - \sum_{i,j=1}^n \left(\frac{\partial a^\varepsilon_{ij}}{\partial x_j} + \frac{\partial a^\varepsilon_{ij}}{\partial \varphi^\varepsilon}\varphi^\varepsilon_{x_j}\right)\varphi^\varepsilon_{x_i} \equiv \overline{B}^\varepsilon(x,\varphi^\varepsilon,D\varphi^\varepsilon).
$$

We take test functions

$$
(A.12) \qquad v^\varepsilon = \sinh\left[\alpha(\varphi^\varepsilon - \varphi)\right]
$$

where $\alpha > 0$ is to be determined. We insert (A.12) into (A.10) and compute

$$
\begin{aligned}
(A.13) \qquad & \alpha \int_\Omega \sum_{i,j=1}^n \zeta a^\varepsilon_{ij}(x,\varphi^\varepsilon)\varphi^\varepsilon_{x_i}(\varphi^\varepsilon-\varphi)_{x_j}\cosh\left(\alpha(\varphi^\varepsilon-\varphi)\right)dx \\
& \quad + \int v^\varepsilon \sum_{i,j=1}^n a^\varepsilon_{ij}(x,\varphi^\varepsilon)\varphi^\varepsilon_{x_i}\zeta_{x_j}\,dx = \int_\Omega \overline{B}^\varepsilon(x,\varphi^\varepsilon,D\varphi^\varepsilon)\zeta v^\varepsilon dx.
\end{aligned}
$$

We further find

$$
\begin{aligned}
& \alpha \int_\Omega \zeta \cosh\left[\alpha(\varphi^\varepsilon-\varphi)\right] \sum_{i,j=1}^n a^\varepsilon_{ij}(x,\varphi^\varepsilon)(\varphi^\varepsilon-\varphi)_{x_i}(\varphi^\varepsilon-\varphi)_{x_j}\,dx \\
(A.14) \qquad & = -\alpha \int_\Omega \zeta \cosh\left[\alpha(\varphi^\varepsilon-\varphi)\right] \sum_{i,j=1}^n a^\varepsilon_{ij}(x,\varphi^\varepsilon)\varphi_{x_i}(\varphi^\varepsilon-\varphi)_{x_j}\,dx \\
& \quad - \int v^\varepsilon \sum_{i,j=1}^n a^\varepsilon_{ij}(x,\varphi^\varepsilon)\varphi^\varepsilon_{x_i}\zeta_{x_j}\,dx + \int_\Omega \overline{B}^\varepsilon(x,\varphi^\varepsilon,D\varphi^\varepsilon)\zeta v^\varepsilon dx.
\end{aligned}
$$

We estimate the middle two integrals in (A.14). We note that $(\varphi^\varepsilon - \varphi)_{x_j} \rightharpoonup 0$ in L^2 weakly by (A.8) for each j. We also note that the term $\cosh\left[\alpha(\varphi^\varepsilon-\varphi)\right] \cdot a^\varepsilon_{ij}(x,\varphi^\varepsilon)$ is bounded and converges almost everywhere as $\varepsilon \to 0+$ for each

(i, j) by the bound (A.5), the smoothness assumption on a^ε_{ij}, and the almost everywhere convergence of φ^ε derived from (A.7). Thus the term $\cosh\left[\alpha(\varphi^\varepsilon - \varphi)\right] a^\varepsilon_{ij}(x, \varphi^\varepsilon)\varphi_{x_i}$ converges strongly in L^2. So the second integral in (A.14) is a sum of just right pairings of weak and strong convergent sequences, and thus converges to zero. We reason similarly on the third integral in (A.14), using $v^\varepsilon = \sinh\left[\alpha(\varphi^\varepsilon - \varphi)\right] \to 0$ strongly, to conclude that it converges to zero, too. Using the ellipticity condition (A.2), the boundedness conditions (A.3) (A.5), and the convergence of the middle two integrals, we deduce

$$(A.15)\qquad \begin{aligned} &\alpha \int_\Omega \zeta \cosh\left(\alpha(\varphi^\varepsilon - \varphi)\right) \lambda(\varphi^\varepsilon)|D(\varphi^\varepsilon - \varphi)|^2 dx \\ &\qquad \le o_\varepsilon(1) + C_M \int_\Omega \zeta(1 + |D\varphi^\varepsilon|^2)|\sinh\left[\alpha(\varphi^\varepsilon - \varphi)\right]| dx, \end{aligned}$$

where $o_\varepsilon(1)$ denotes a quantity that goes to zero as $\varepsilon \to 0+$ for any fixed α. Replacing $|D\varphi^\varepsilon|^2$ by $2|D(\varphi^\varepsilon - \varphi)|^2 + 2|D\varphi|^2$, we further deduce

$$(A.16)\qquad \begin{aligned} &\alpha \int_\Omega \zeta \cosh\left(\alpha(\varphi^\varepsilon - \varphi)\right) \lambda(\varphi^\varepsilon)|D(\varphi^\varepsilon - \varphi)|^2 dx \\ &\qquad \le o_\varepsilon(1) + 2C_M \int_\Omega \zeta|D(\varphi^\varepsilon - \varphi)|^2 \cosh\left(\alpha(\varphi^\varepsilon - \varphi)\right) dx \end{aligned}$$

where we once again used the pairing of weak-strong convergent sequences and the simple fact $|\sinh(s)| \le \cosh(s)$ $(s \in \mathbb{R})$. Let $\Omega'' \subset \Omega$ be the support of ζ. Then $\varphi^\varepsilon(x) \ge \xi(x) > \xi_{\min}$ in Ω'' for a positive number $\xi_{\min} > 0$. Thus $\lambda(\varphi^\varepsilon) \ge \lambda_{\min} > 0$. So we have from (A.16)

$$(\alpha\lambda_{\min} - 2C_M) \int_{\Omega''} \zeta \cosh\left(\alpha(\varphi^\varepsilon - \varphi)\right) |D(\varphi^\varepsilon - \varphi)|^2 dx \le o_\varepsilon(1).$$

Observing that $\Omega' \subset \Omega''$ and $\cosh(s) \ge 1$ $(s \in \mathbb{R})$, we finally obtain

$$(\alpha\lambda_{\min} - 2C_M) \int_{\Omega'} |D(\varphi^\varepsilon - \varphi)|^2 dx \le o_\varepsilon(1).$$

Adjusting α to be large, e.g., $\alpha\lambda_{\min} = 2C_M + 1$, we deduce the strong convergence of $D\varphi^\varepsilon$ to $D\varphi$ in $L^2(\Omega')$.

ACKNOWLEDGEMENTS

The author wishes to thank L. C. Evans, Taiping Liu, and Tong Zhang for stimulating discussions and suggestions. The author also wishes to thank a referee for his/her questions on an earlier presentation of the appendix and for calling his attention to a couple of references. This work is supported in part by NSF DMS–9303414 and an Alfred P. Sloan Research Fellows award.

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