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Two-Dimensional Riemann Problems for the Compressible Euler System**

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Abstract Riemann problems for the compressible Euler system in two space dimensions are complicated and difficult, but a viable alternative remains missing. We list merits of one-dimensional Riemann problems and compare them with those for the current two-dimensional Riemann problems, to illustrate their worthiness. We approach two-dimensional Riemann problems via the methodology promoted by Andy Majda in the spirits of modern applied mathematics; that is, simplified model building via asymptotic analysis, numerical simulation, and theoretical analysis. We derive a simplified model, called the pressure gradient system, from the full Euler system via an asymptotic process. We use state-of-the-art numerical methods in numerical simulations to discern small-scale structures of the solutions, e.g., semi-hyperbolic patches. We use analytical methods to establish the validity of the structure revealed in the numerical simulation. The entire process, used in many of Majda's programs, is shown here for the two-dimensional Riemann problems for the compressible Euler systems of conservation laws.

Keywords Characteristic decomposition, Guderley reflection, hodograph transform, pressure gradient system, self-similar, semi-hyperbolic wave, triple point paradox, Riemann problem, Riemann variable.
2000 MR Subject Classification Primary: 35L65, 35J70, 35R35; Secondary: 35J65.

1 Systems

We consider the two-dimensional (2-D) compressible Euler system

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + pI) = 0, \\ (\rho E)_t + \nabla\cdot(\rho E\mathbf{u} + p\mathbf{u}) = 0, \end{cases} \tag{1.1}$$

where ρ is density, $\mathbf{u}$ is velocity vector, p is pressure, $E = |\mathbf{u}|^2/2 + \gamma p/\rho$ is the total energy density, and $\gamma > 1$ is the gas constant. We also consider the so-called **pressure gradient system**

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ E_t + (up)_x + (vp)_y = 0, \end{cases} \tag{1.2}$$

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where $E = p + (u^2 + v^2)/2$. Cauchy problems for both systems are open.

2 Search for model.

The full Euler system is known to be hard. Simplified models are appreciated. We propose to asymptotically replace the Euler by

$$\begin{cases} \rho & \longrightarrow & 1 & + & \frac{1}{\gamma}\rho, \\ \mathbf{u} & \longrightarrow & & & \frac{1}{\gamma}\mathbf{u}, \\ p & \longrightarrow & & & \frac{1}{\gamma}p \end{cases} \tag{2.1}$$

in the limit $\gamma \to \infty$. To leading orders of the equations we obtain the pressure gradient system.

In the asymptotic process we note that the sound speed

$$c = \sqrt{\gamma p/\rho} \sim \sqrt{p}$$

remains at order one, thus our asymptotic system catches acoustic waves.

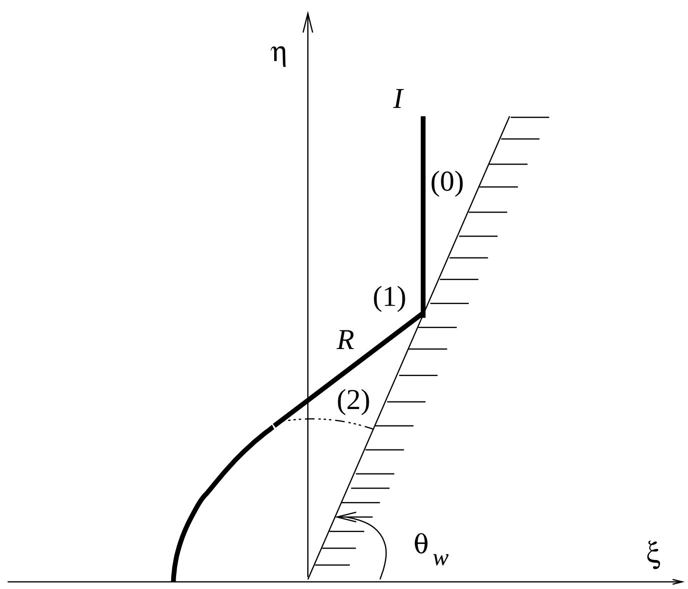

Figure 1. Regular reflection

The pressure decouples from the velocity field to form

$$\left(\frac{p_t}{p}\right)_t - \Delta p = 0$$

which is indeed simpler than the Euler system. The pressure gradient system is the first two-dimensional system of two or more equations to have the existence established of a global regular reflection on a wedge, see Figure 1 and the reference Y. Zheng [42]. Other progresses are [1, 6, 11, 14, 31, 32, 40, 41].

3 Riemann problems: 2-D

We propose Riemann problems as initial-value problems in which the initial values are independent of the spatial radius $r = \sqrt{x^2 + y^2}$, $(x, y) \in \mathbb{R}^2$. A typical realization is to have four constants instead of an arbitrary function of the polar angle θ, see Figure 2.

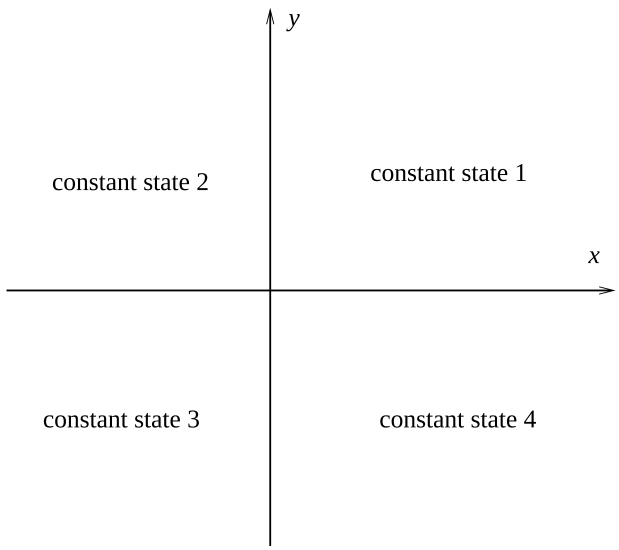

Figure 2: Four−constant Riemann problem.

The most important feature of the Riemann problems is that we can propose to look for the so-called *self-similar* solutions that depend only on the variables $\xi = \frac{x}{t}$, $\eta = \frac{y}{t}$. In this regard, we include some initial-boundary value problems, such as a planar shock hitting a straight wedge, as Riemann problems, as long as the solutions are self-similar.

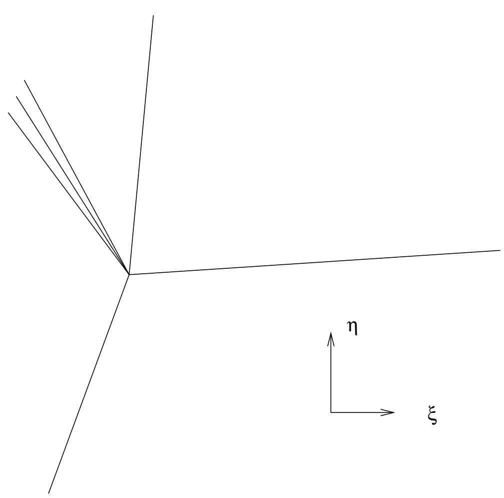

Figure 3. Elementary Waves.

There are other possibilities for Riemann problems. For example, one may search for *elementary waves* (rather than working with given initial data) as Riemann problems, as James Glimm and associates once attempted, see Figure 3. By elementary waves, we mean waves that will make up general solutions.

To judge 2-D Riemann problems, let us list the merits of 1-D Riemann problems:

1. Building blocks, (for solutions to Cauchy problems)
2. Asymptotic states, (of solutions as time approaches infinity)
3. Elementary waves,
4. Simple and easy.

And here are features of our 2-D Riemann problems:

1. Complicated and hard.
2. Their solutions reveal a lot of internal structures of solutions.
3. Their solutions depend only on two independent variables.
4. They can be used to build general approximate solutions.

So it is clear that an ideal generalization of the one-dimensional Riemann problem is not available, the current 2-D Riemann problem is below expectation, but it is still valuable to use to get into two-dimensional problems.

4 Early results

There are long-standing interests in multi-dimensional piecewise smooth solutions, see e.g. [8, 25, 27]. In 1986, the four-constant two-dimensional Riemann problem for a typical scalar conservation law was solved, see [34, 37]. For the full Euler, a set of educated guesses of solutions to the four-wave two-dimensional Riemann problems was proposed in 1990 [38]. (The four-wave two-dimensional Riemann problems are special cases of the four-constant two-dimensional Riemann problems.) Assuming axial symmetry, the two-dimensional Riemann problems were solved for the Euler in 1996 [39], see also [44]. The axial case catches solutions that exhibit structures of an eye with an eye-wall that are important features of a hurricane.

We would like to mention an interesting structure of a solution to the two-dimensional Riemann problem for the scalar conservation law

$$u_t + (u^2/2)_x + (u^3/3)_y = 0$$

from Guckenheimer [9], sketched in Figure 4. The initial data is a triple discontinuity meeting at one point, but the solution does not keep the simple structure; instead, the triple point exhibits the so-called *von Neumann structure* with a small rarefaction wave (R). In Figure 4, S(I) is interpreted as incident shock, S(M) as Mach stem, while S(R) as reflected shock in analogy to the Mach reflection in a planar shock hitting a wedge.

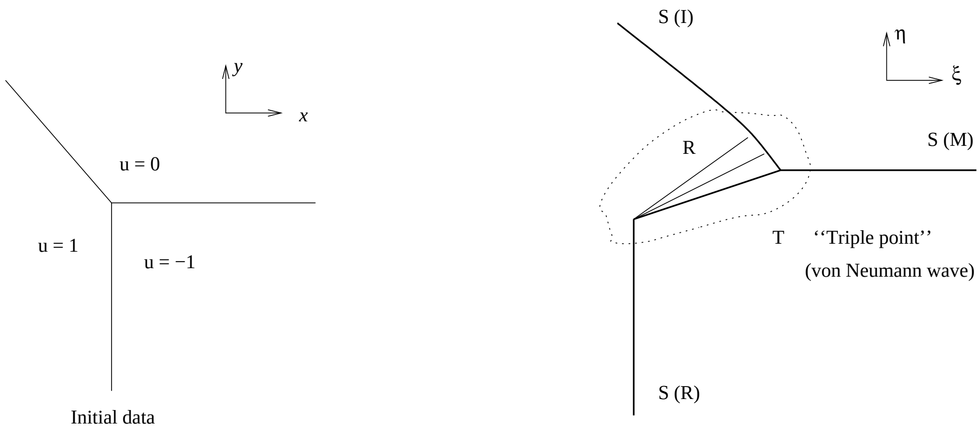

Figure 4. Guckenheimer solution.

This brings up Euler system's von Neumann triple point paradox, for which John Hunter and collaborators have produced numerical evidence of Guderley reflection [10, 35, 36]. See Skews and Ashworth [30] for physical experimental evidence. An illustration is shown in Fig. 5.

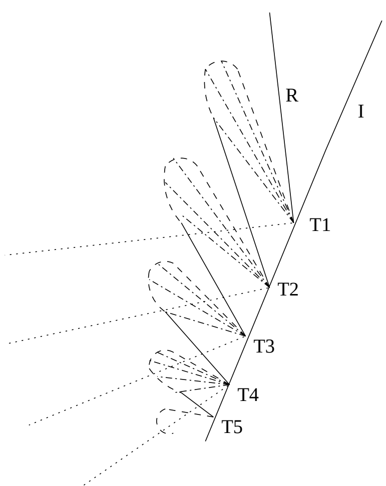

Figure 5: Guderley reflection

In addition, we find that the 4R interaction (two forward, two backward) of the two-dimensional Riemann problems contains patches of solutions adjacent to sonic curves, that are like the small waves in the Guderley reflection. See Figure 6, which is from [7]. These small patches are hyperbolic, but one family of characteristics fails to connect to infinity. So they will be called *semi-hyperbolic.* Other similar numerical results are [2, 12, 13, 29].

We are interested in constructing the pieces of semi-hyperbolic waves adjacent to the sonic curves. To place the semi-hyperbolic waves in perspective, we notice three wave types:

1. Simple waves,
2. Interactions of binary planar waves,

3. Semi-hyperbolic waves.

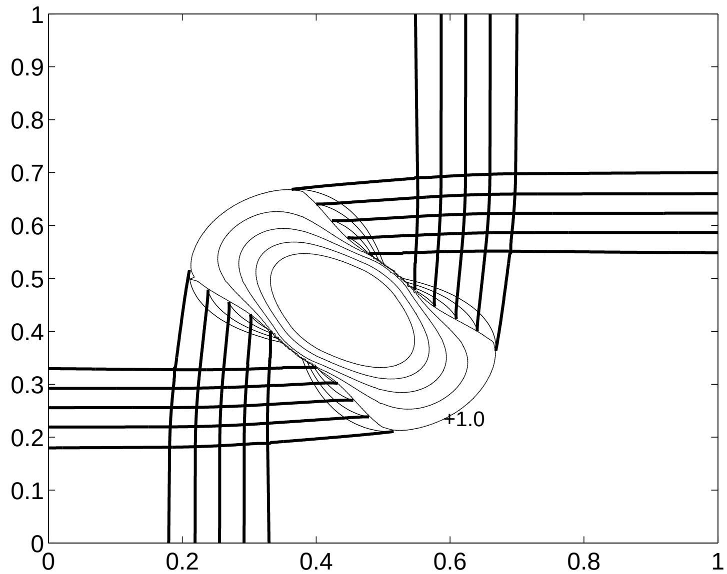

Figure 6: Four semi-hyperbolic patches (netted by thin and heavy curves).

A simple wave occupies a region in which one family of characteristics are all straight lines. A planar wave is a simple wave in which one family of characteristics are straight and parallel. Interaction of two planar waves is typically not a simple wave anymore. Semi-hyperbolic waves are locally hyperbolic, but one family of characteristics all start on and end on the sonic curve (or a transonic shock wave). The wave fans in the Guderley reflection of Figure 5 and the four patches in the 4R interaction of Figure 6 are semi-hyperbolic.

5 Self–similar Euler: Simple waves

We develop analytic methods. First we write the self-similar form of the Euler system:

$$\begin{cases} (u-\xi)i_\xi + (v-\eta)i_\eta + 2\kappa\, i\,(u_\xi + v_\eta) = 0, \\ (u-\xi)u_\xi + (v-\eta)u_\eta + i_\xi = 0, \\ (u-\xi)v_\xi + (v-\eta)v_\eta + i_\eta = 0, \end{cases} \tag{5.1}$$

where $i = c^2/(\gamma-1), \kappa = (\gamma-1)/2$. The two nonlinear eigenvalues are

$$\Lambda_\pm = \frac{(u-\xi)(v-\eta) \pm c\sqrt{(u-\xi)^2 + (v-\eta)^2 - c^2}}{(u-\xi)^2 - c^2}. \tag{5.2}$$

For catching the simple waves, we formulate the identities, see [21]:

$$\begin{cases} \partial^+(\partial^- u) + \dfrac{\partial^+\Lambda_- - \partial^-\Lambda_+}{\Lambda_+ - \Lambda_-}\partial^- u = \dfrac{\Lambda_+\Lambda_-}{\Lambda_+ - \Lambda_-}\left[\dfrac{\partial^-\Lambda_-}{\Lambda_-^2}\partial^+ u - \dfrac{\partial^+\Lambda_+}{\Lambda_+^2}\partial^- u\right], \\ \partial^-(\partial^+ u) + \dfrac{\partial^+\Lambda_- - \partial^-\Lambda_+}{\Lambda_+ - \Lambda_-}\partial^+ u = \dfrac{\Lambda_+\Lambda_-}{\Lambda_+ - \Lambda_-}\left[\dfrac{\partial^-\Lambda_-}{\Lambda_-^2}\partial^+ u - \dfrac{\partial^+\Lambda_+}{\Lambda_+^2}\partial^- u\right]; \end{cases}$$

$$\partial^{\pm}\Lambda_{\pm} = \left[\partial_U\Lambda_{\pm} - \Lambda_{\mp}^{-1}\partial_V\Lambda_{\pm} - (\gamma-1)\partial_{c^2}\Lambda_{\pm}(U - \Lambda_{\mp}^{-1}V)\right]\partial^{\pm}u.$$

The notations are $\partial^{\pm} = \partial_\xi + \Lambda_{\pm}\partial_\eta, U = u - \xi, V = v - \eta$, and $\Lambda_{\pm}$ are regarded as functions of three independent variables (U, V, c^2) in $\partial_U\Lambda_{\pm}, \partial_V\Lambda_{\pm}$, and $\partial_{c^2}\Lambda_{\pm}$.

We then see that $\partial^- u = 0$ along an entire curve of plus characteristics, provided that it is zero at any point at all. This way, we conclude that a wave that is adjacent to a constant state must be a simple wave.

6 Main approaches:

We note that we can approach the theoretical construction of solutions via two directions:

1. Hodograph transform;
2. Direct characteristics decomposition.

We recall the classical hodograph transform

$$(x, y) \longrightarrow (u, v)$$

is for a homogeneous system like this

$$\begin{cases} u_x + a(u,v)u_y + b(u,v)v_y &= 0, \\ v_x + c(u,v)u_y + d(u,v)v_y &= 0. \end{cases} \tag{6.1}$$

And the system in the (u, v) plane is linear.

For possible hodograph transform for the self–similar Euler (5.1) we again use $(\xi, \eta) \to (u, v)$ and regard i as a function of (u, v): $i = i(u, v)$. It brings the system to a single equation

$$(c^2 - i_u^2)i_{vv} + 2i_u i_v i_{uv} + (c^2 - i_v^2)i_{uu} = i_u^2 + i_v^2 - 2c^2 \tag{6.2}$$

in the hodograph plane. This equation is not linear, but it is linearly degenerate.

To show it, we introduce the *inclination angles* of characteristics (α, β):

$$\tan\alpha = \Lambda_+; \qquad \tan\beta = \Lambda_-.$$

Note that $\omega = (\alpha - \beta)/2$ is the so-called (pseudo-)Mach angle, see Courant and Friedrichs [5]. The system in the hodograph plane is

$$\begin{cases} \bar\partial_+\alpha &= \dfrac{1+\gamma}{4c}\cdot\sin(\alpha-\beta)\cdot\left(m - \tan^2\omega\right), \\ \bar\partial_-\beta &= \dfrac{1+\gamma}{4c}\cdot\sin(\alpha-\beta)\cdot\left(m - \tan^2\omega\right), \\ \partial_0 c &= \kappa\dfrac{\cos\frac{\alpha+\beta}{2}}{\sin\omega} \end{cases} \tag{6.3}$$

with

$$\bar\partial_+ c = -\kappa, \quad \bar\partial_- c = \kappa.$$

where,

$$\bar\partial_+ = \sin\beta\partial_u - \cos\beta\partial_v, \quad \bar\partial_- = \sin\alpha\partial_u - \cos\alpha\partial_v, \quad \partial_0 = \partial_u,$$

and

$$m = \frac{3-\gamma}{1+\gamma}.$$

It is linearly degenerate since the variable α is differentiated along a direction determined by β.

We have obtained other identities to handle high-order estimates and the one-to-one correspondence between the self-similar plane and the hodograph plane, see paper [22].

7 Application 1: Two rarefaction wave interaction

It is also called a wedge of gas expanding into vacuum, see Figure 7.

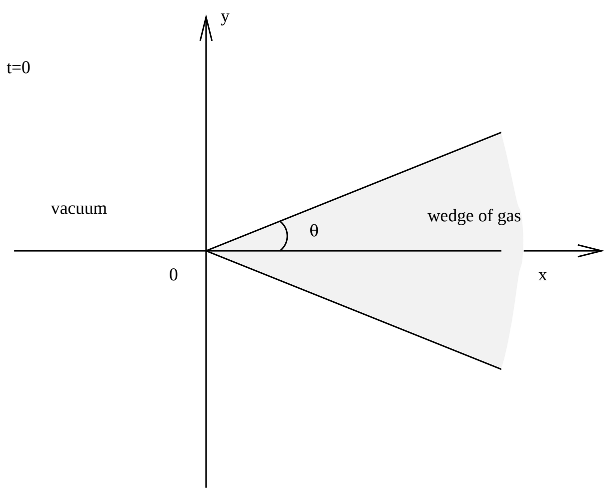

Figure 7: Initial set-up of a wedge of gas expanding into vacuum.

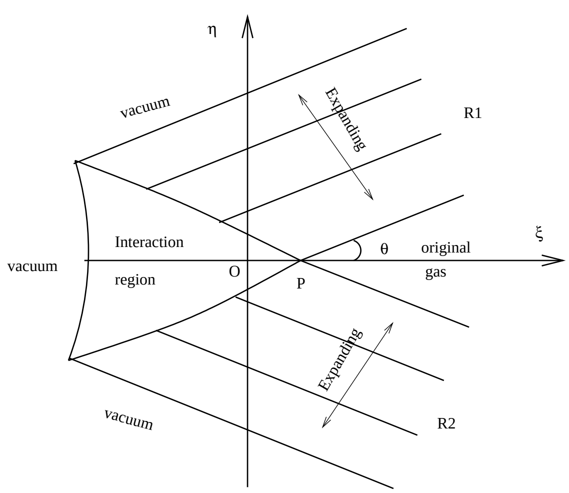

Figure 8: Interaction position of a wedge of gas expanding into vacuum.

It was considered by Suchkov in 1963 [33] and Levine and Mackie in 1968 [15, 26]. The interaction zone is illustrated in Figure 8.

Wave interaction region in the hodograph plane is illustrated in Figure 9.

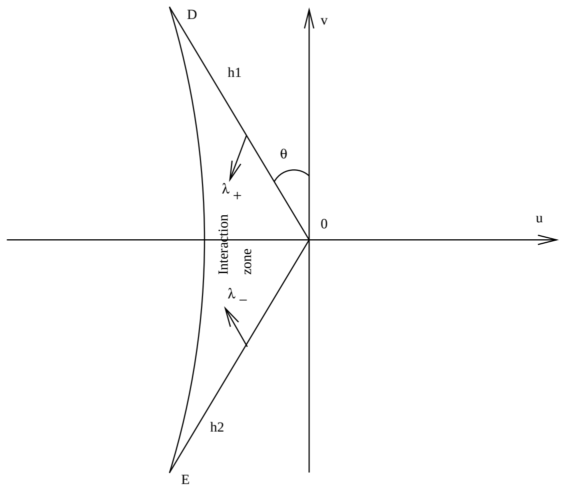

Figure 9: Interaction zone in the hodograph plane.

We obtain the solutions, one typical solution is shown in Figure 10, where θ_s is defined by $\tan^2\theta_s = m$.

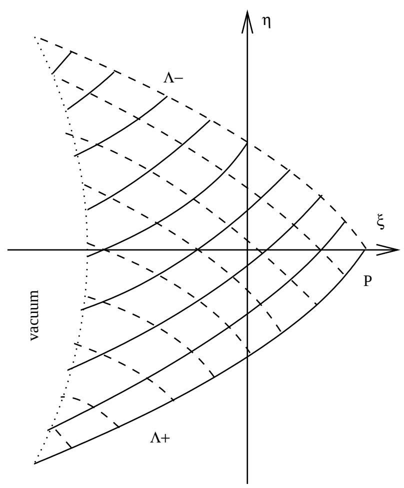

Figure 10: $\theta_s < \theta < 2\theta_s$

These results are from the paper with Jiequan Li, see [22].

Why does it work well here, but not well in the steady case? The relation

$$\xi = u + i_u, \qquad \eta = v + i_v$$

is different from the steady case:

$$0 = u + i_u, \qquad 0 = v + i_v.$$

The hodograph transform for (5.1) was carried out a while ago in Pogodin, Suchkov and Ianenko, see [28], but it seems it has not been used well until recently, see Li [16, 17].

8 Direct method

We want to by-pass the hodograph transform. We obtain ([4, 19])

$$\begin{cases} \bar{\partial}^+(-\beta+\psi(\omega)) = \dfrac{\sin^2\omega[\cos(2\omega)-\kappa]}{c(\kappa+\sin^2\omega)}, \\ \bar{\partial}^-(\alpha+\psi(\omega)) = \dfrac{\sin^2\omega[\cos(2\omega)-\kappa]}{c(\kappa+\sin^2\omega)}, \\ \bar{\partial}^0[c^2(1+\kappa M^2)] = 2c\kappa M, \end{cases} \tag{8.1}$$

where $M^2 = (u^2+v^2)/c^2$, $\bar{\partial}^+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta, \bar{\partial}^- = \cos\beta\partial_\xi + \sin\beta\partial_\eta, \bar{\partial}^0 := -\cos\zeta\partial_\xi - \sin\zeta\partial_\eta, \zeta = (\alpha+\beta)/2$ and

$$\psi(\omega) := \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\cot\omega\right). \tag{8.2}$$

The *Riemann variables* $\psi-\beta$ and $\psi+\alpha$ correspond to the classical Riemann invariants for homogeneous systems.

Applying the direct method to the expansion of a wedge of gas, we obtain the same conclusion as before. The computations are not easier than the hodograph method, though.

9 Semi-hyperbolic patches (Application 2)

Next in line is the semi-hyperbolic patch. A semi-hyperbolic patch has one family of characteristics that start and end on sonic curves or transonic shocks. They are abound, see Fig. 6.

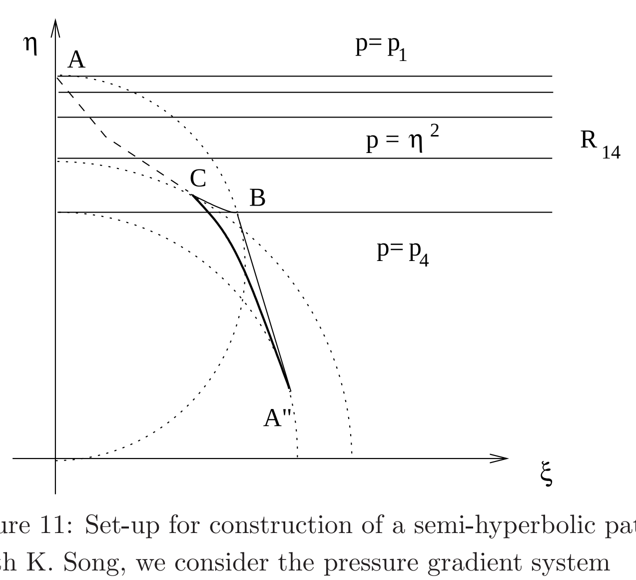

Figure 11: Set-up for construction of a semi-hyperbolic patch.

In paper [32] with K. Song, we consider the pressure gradient system

$$p_{\theta\theta} - \frac{r^2(r^2-p)}{p}p_{rr} + \frac{rp_r}{p}(p + \frac{r^3p_r}{p} - 2r^2) = 0$$

in self-similar polar coordinates (r, θ). It decomposes to

$$\begin{aligned} \partial^+\partial^- p &= q \cdot (\partial^+ p - \partial^- p) \cdot \partial^- p, \\ \partial^-\partial^+ p &= q \cdot (\partial^- p - \partial^+ p) \cdot \partial^+ p, \end{aligned} \tag{9.1}$$

where $q := \frac{r^2}{4p(r^2-p)}$. Here $\partial^\pm$ are again derivatives along characteristics. We consider the set-up as in Figure 11. The horizontal planar wave $p = \eta^2$ is given up to the boundary AB, and the curve BC is given as a convex characteristic curve of the minus family, with point C being sonic. The characteristics in the domain ABC are drawn in Figure 12.

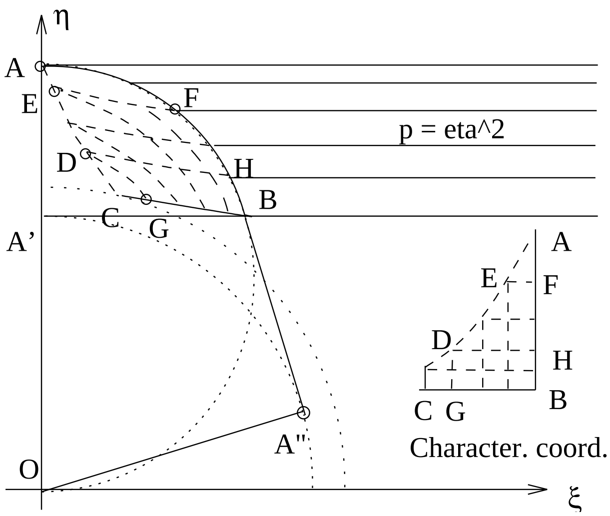

Figure 12: Solution for a semi-hyperbolic patch AFHBGCDEA.

Maximum principle holds for $\partial^\pm p$ in the semi-hyperbolic region. A cute proof using Figure 13 is in paper [32].

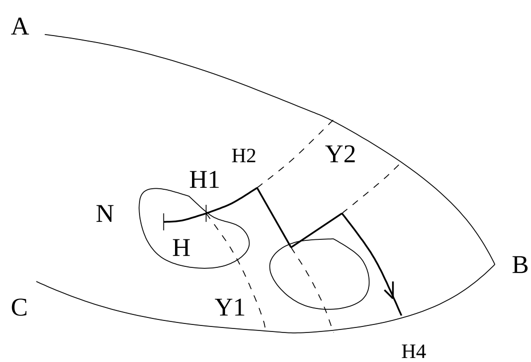

Figure 13: Establishing the maximum principle.

Envelope forms in the simple wave region, thus shock is present, see Figure 14. Thus the patch $ABA"CA$ is semi-hyperbolic.

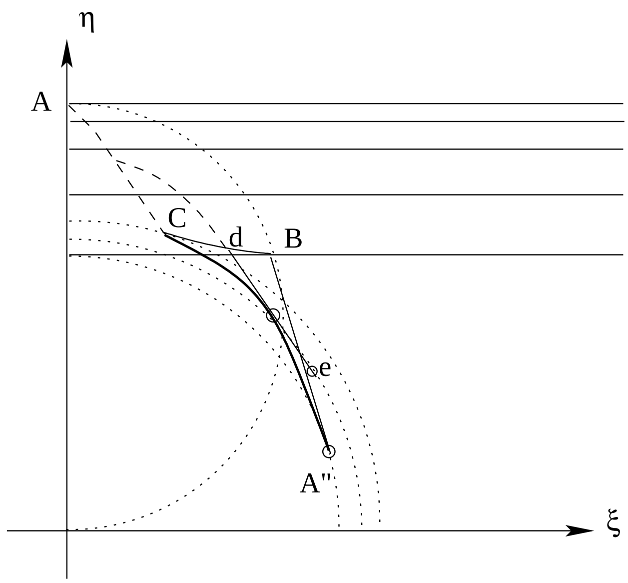

Figure 14: Envelope forms along CA".

We have the same conclusion for the Euler, see paper with Mingjie Li [24]. Note the new decomposition:

$$\begin{cases} c\bar{\partial}^+\Big(\dfrac{\bar{\partial}^- c}{\sin^2\omega}\Big) + \Big[\dfrac{8\sin^2\omega}{\gamma-1}+2\Big]\bar{\partial}^+ c\Big(\dfrac{\bar{\partial}^- c}{\sin^2\omega}\Big) = \dfrac{\nu}{\cos^2\omega}\Big(\dfrac{\bar{\partial}^- c}{\sin^2\omega}\Big)\Big[\bar{\partial}^+ c + \bar{\partial}^- c\Big], \\ c\bar{\partial}^-\Big(\dfrac{\bar{\partial}^+ c}{\sin^2\omega}\Big) + \Big[\dfrac{8\sin^2\omega}{\gamma-1}+2\Big]\bar{\partial}^- c\Big(\dfrac{\bar{\partial}^+ c}{\sin^2\omega}\Big) = \dfrac{\nu}{\cos^2\omega}\Big(\dfrac{\bar{\partial}^+ c}{\sin^2\omega}\Big)\Big[\bar{\partial}^- c + \bar{\partial}^+ c\Big]. \end{cases}$$

In the interaction of 4R case, we note that one simple case occurs when the central subsonic region degenerates to vacuum so that no semi-hyperbolic wave is reflected, and a global continuous solution is obtained, see paper with Jiequan Li [23].

10 Summary

We bring new life to the hodograph method so that it also works for the Euler system of 3×3. We find Riemann variables $\{-\beta+\psi(\omega),\, \alpha+\psi(\omega)\}$ where

$$\psi(\omega) := \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\cot\omega\right)$$

for the 3×3 Euler system. And we build semi-hyperbolic patches of solutions.

These methods and ideas may apply to applications in Mach (Guderley) reflection, channel flow, flow around airfoil, de Laval nozzle, etc., see e.g. Chen [3].

With regards to numerics: Theoretical work and numerics need this mutual movement – challenge and promote each other.

See the survey paper [18] for more details. See the books [20, 43] for more background. We apologize for not able to cover work of Canic, Keyfitz, Kim, Tesdall, Hunter, Guiqiang Chen, Feldman, T. P. Liu, V. Elling, Shuxing Chen, Huicheng Yin, Zhouping Xin, Denis Serre, et. al. in more details.

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