

RAMANUJAN'S PARTIAL THETA SERIES AND PARITY IN PARTITIONS

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Dedicated to George Andrews for his 70th birthday

Abstract. A partial theta series identity from Ramanujan's lost notebook has a connection with some parity problems in partitions studied by Andrews in [3], where 15 open problems are listed. In this paper, the partial theta series identity of Ramanujan is revisited and answers to Questions 9 and 10 of Andrews are provided.

1. INTRODUCTION

In his recent paper [3], George Andrews investigated a variety of parity questions in partition identities. At the end of the paper, he then listed 15 open problems, two of which have a connection with the following partial theta series from Ramanujan's lost notebook.

Theorem 1.1. [8, p. 28], [4, Entry 1.6.2] *For any complex number a ,*

$$1 + \sum_{n=1}^{\infty} \frac{(-q; q)_{n-1} a^n q^{n(n+1)/2}}{(-aq^2; q^2)_n} = \sum_{n=0}^{\infty} a^n q^{n^2}. \quad (1.1)$$

As customary, here and in the sequel, we employ the standard notation

$$(a; q)_0 = 1, \quad (a; q)_n := (1-a)(1-aq) \cdots (1-aq^{n-1}), \quad n \geq 1,$$

and

$$(a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n, \quad |q| < 1.$$

For a partition λ , we define $I_{UO}(\lambda)$ (resp. $I_{UE}(\lambda)$) by the maximum length of weakly decreasing subsequences of the parts whose terms alternate in parity starting with an odd (resp. even) part.

Example 1. $\lambda = (7, 7, 5, 4, 4, 3, 2, 1, 1)$. Then

$$I_{UO}(\lambda) = 5, \quad I_{UE}(\lambda) = 4.$$

Let $\delta_o(N, r, m, n)$ (resp. $\delta_e(N, r, m, n)$) denote the number of partitions of n into m distinct parts $\leq N$ with upper odd (resp. even) parity index equal to r . We define

$$D_o(N, y, x; q) := D_o(N) = \sum_{r, m, n \geq 0} \delta_o(N, r, m, n) y^r x^m q^n,$$
$$D_e(N, y, x; q) := D_e(N) = \sum_{r, m, n \geq 0} \delta_e(N, r, m, n) y^r x^m q^n.$$

Then we have the following theorem.

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Theorem 1.2 (Andrews [3]).

$$D_o(\infty) = \sum_{i,j \geq 0} \frac{x^i y^{2j} q^{(i-j)^2 + j^2 + i + j}}{(-q; q)_i (q)_{2j} (q)_{i-2j}} + \sum_{i,j \geq 0} \frac{x^i y^{2j-1} q^{(i-j)^2 + j^2 + i - j}}{(-q; q)_i (q)_{2j-1} (q)_{i-2j+1}}, \quad (1.2)$$

$$D_e(\infty) = \sum_{i,j \geq 0} \frac{x^i y^{2j} q^{(i-j)^2 + j^2 + j}}{(-q; q)_i (q)_{2j} (q)_{i-2j}} + \sum_{i,j \geq 0} \frac{x^i y^{2j+1} q^{(i-j)^2 + j^2 + 3j + 1}}{(-q; q)_i (q)_{2j+1} (q)_{i-2j-1}}. \quad (1.3)$$

The ninth and tenth questions of Andrews are as follows.

Question 9. Prove that if $x = -1$ and $y = 1$, then the second sum of (1.2) is

$$\sum_{n=1}^{\infty} (-1)^n q^{n^2}.$$

Question 10. Prove that if $x = -1$ and $y = 1$, then the first sum of (1.3) is

$$\sum_{n=0}^{\infty} (-1)^n q^{n^2}.$$

The primary purpose of this paper is to provide answers to these two questions of Andrews, which involve Ramanujan's partial theta series (1.1).

In section 2, we give answers to Andrews' questions. Then, a combinatorial proof of Theorem 1.1 will be given in Section 3. Theorem 1.2 will be proved combinatorially in Section 4. In the last section, some remarks are made, in particular, a sketch of a solution to Question 5 of Andrews [3] is given.

2. OPEN QUESTIONS 9 AND 10 OF ANDREWS

In this section, we will provide answers to his two open questions. We first need the following lemma.

Lemma 2.1. For any positive integer n ,

$$\sum_{k=0}^n q^{(n-k)^2 + k^2 + n - k} \begin{bmatrix} n \\ 2k-1 \end{bmatrix}_q = \sum_{k=0}^n q^{(n-k)^2 + k^2 + k} \begin{bmatrix} n \\ 2k \end{bmatrix}_q = (-q; q)_{n-1} q^{n(n+1)/2}, \quad (2.1)$$

where

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} \frac{(q; q)_a}{(q; q)_b (q; q)_{a-b}}, & \text{if } 0 \leq b \leq a, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We will only show that

$$\sum_{k=0}^n q^{(n-k)^2 + k^2 + k} \begin{bmatrix} n \\ 2k \end{bmatrix}_q = (-q; q)_{n-1} q^{n(n+1)/2},$$

which is equivalent to

$$\sum_{k=0}^n q^{(n-2k)(n-2k-1)/2} \begin{bmatrix} n \\ 2k \end{bmatrix}_q = (-q; q)_{n-1}.$$

This follows from the recurrence of the q -binomial coefficients, namely

$$\begin{aligned} \sum_{k=0}^n q^{(n-2k)(n-2k-1)/2} \begin{bmatrix} n \\ 2k \end{bmatrix}_q &= \sum_{k=0}^n q^{(n-2k)(n-2k-1)/2} \left(\begin{bmatrix} n-1 \\ n-2k-1 \end{bmatrix}_q + q^{n-2k} \begin{bmatrix} n-1 \\ n-2k \end{bmatrix}_q \right) \\ &= \sum_{k=0}^{n-1} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \\ &= (-q; q)_{n-1}. \end{aligned}$$

□

We now give answers to Questions 9 and 10 in the following theorem.

Theorem 2.2. *We have*

$$\begin{aligned} \sum_{n,k \geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+n-k}}{(-q; q)_n (q)_{2k-1} (q)_{n-2k+1}} &= \sum_{n=1}^{\infty} (-1)^n q^{n^2}, \\ \sum_{n,k \geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+k}}{(-q; q)_n (q)_{2n} (q)_{n-2k}} &= \sum_{n=0}^{\infty} (-1)^n q^{n^2}. \end{aligned}$$

Proof. Rewrite the left-hand side of the first identity as

$$\begin{aligned} \sum_{n,k \geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+n-k}}{(-q; q)_n (q)_{2k-1} (q)_{n-2k+1}} &= \sum_{i,j \geq 0} \frac{(-1)^n q^{(n-k)^2+k^2+n-k}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k-1 \end{bmatrix}_q \\ &= \sum_{n \geq 1} \frac{(-1)^n (-q; q)_{n-1} q^{n(n+1)/2}}{(q^2; q^2)_n}, \end{aligned}$$

where the second equality follows from Lemma 2.1. Setting $a = -1$ in (1.1), we arrive at

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n (-q; q)_{n-1} q^{n(n+1)/2}}{(q^2; q^2)_n} = \sum_{n=0}^{\infty} (-1)^n q^{n^2}.$$

which completes the proof.

Since the proof of the second identity is similar, we omit it. □

3. COMBINATORICS OF RAMANUJAN'S PARTIAL THETA SERIES

In this section, we provide a combinatorial proof of the partial theta series identity (1.1) from Ramanujan's lost notebook.

We introduce some terminology needed in the rest of this paper. For a partition λ , $\ell(\lambda)$ denotes the number of positive parts, and we define $\lambda_i = 0$ for $i > \ell(\lambda)$. We also denote by \emptyset the empty partition of 0. For partitions λ and μ , we define the sum $\lambda + \mu$ of λ and μ to be the partition whose i th part is $\lambda_i + \mu_i$. We denote the conjugate of a partition λ by λ' (see [2].)

Theorem 3.1. *For any positive integer n , the generating function of partitions λ into n distinct parts with $\lambda_i - \lambda_{i+1} \leq 2$ and even upper even parity index is*

$$(-q; q)_{n-1} q^{n(n+1)/2}.$$

Note that the least part of the partitions λ stated in Theorem 3.1 must be 1 since $\lambda_{\ell(\lambda)} \leq 2$ and their even upper parity index is even.

Proof. Let $\tau = (n, n-1, \dots, 1)$ and μ be a partition generated by $(-q; q)_{n-1}$. We add each part of μ to τ vertically from the largest part and denote the resulting partition λ . That is,

$$\lambda = \tau + \mu'.$$

Since the distinct parts of μ are added, the adjacent parts of λ differ by at most 2. This process is reversible. Let i be the smallest integer such that $\lambda_i - \lambda_{i+1} = 2$. We then subtract 1 from each of the largest i parts of λ . Repeating this until there is no such i , i.e., the resulting partition is τ .

We now show that the upper even parity index of the λ is $2\lfloor(n - \ell(\mu))/2\rfloor$. We use induction on the number of parts of μ . If $\mu = \emptyset$, then $\lambda = \tau$, the upper even parity index of which is $2\lfloor n/2\rfloor$. Suppose that μ has k parts, and let λ be the partition resulting from insertion of all parts but the smallest one of μ . Since the insertion process is performed vertically, we see that for $i < \mu_{k-1}$,

$$\lambda_i - \lambda_{i+1} = 1.$$

We also see that vertical insertion of μ_k changes the parity of only the μ_k largest parts of λ . Thus, the μ_k th part and the $(\mu_k + 1)$ th part have the same parity, from which we see that the upper even parity index reduces by 1. Furthermore, if $n + k$ is even, the parity index increases by 1; while if $n + k$ is odd, the parity index decreases by 1. Hence, we see that the parity index is

$$\begin{cases} 2\lfloor(n - k + 1)/2\rfloor, & \text{if } n + k = \text{even,} \\ 2\lfloor(n - k + 1)/2\rfloor - 2, & \text{if } n + k = \text{odd,} \end{cases}$$

which is equivalent to

$$2\lfloor(n - k)/2\rfloor.$$

Therefore, the generating function is

$$(-q; q)_{n-1} q^{n(n+1)/2}.$$

□

Note that a partition with even upper even parity index has odd upper odd parity index, and vice versa. Thus, a similar result on the generating function of partitions into distinct parts and odd upper odd parity index follows from Theorem 3.1. We state this in the following corollary.

Corollary 3.2. *For any positive integer n , the generating function of partitions λ into n distinct parts with $\lambda_i - \lambda_{i+1} \leq 2$, $\lambda_n = 1$, and odd upper odd parity index is*

$$(-q; q)_{n-1} q^{n(n+1)/2}.$$

It follows from Theorem 3.1 and Corollary 3.2 that the generating function of partitions into n distinct parts with even (resp. odd) upper even (resp. odd) parity index is

$$\frac{(-q; q)_{n-1} q^{n(n+1)/2}}{(q^2; q^2)_n}.$$

We now recall Theorem 1.1 with a replaced by $-a$.

Theorem 3.3. [8, p. 28], [4, Entry 1.6.2]

$$1 + \sum_{n=1}^{\infty} \frac{(-a)^n (-q; q)_{n-1} q^{n(n+1)/2}}{(aq^2; q^2)_n} = \sum_{n=0}^{\infty} (-a)^n q^{n^2}. \quad (3.1)$$

Proof. Let $D_e(E)$ be the set of partitions into distinct parts with even upper even parity index. As noted after Corollary 3.2, the left-hand side of (3.1) generates partitions π in $D_e(E)$. We will prove the theorem by setting up a sign reversing involution.

Let λ be a partition into distinct parts with $\lambda_i - \lambda_{i+1} \leq 2$ and even upper even parity index, and let σ be a partition into even parts $\leq 2\ell(\lambda)$. Then it follows from the remark made after Corollary 3.2 that

$$\lambda + \sigma' \in D_e(E).$$

This process is reversible. For a partition $\pi \in D_e(E)$, by taking out

$$2i \lfloor (\pi_i - \pi_{i+1})/2 \rfloor$$

boxes in columns from right to left in the Ferrers graph of π , we can also decompose π into λ and σ , where λ is a partition into distinct parts with $\lambda_i - \lambda_{i+1} \leq 2$ and even upper even parity index, and σ is a partition into even parts $\leq 2\ell(\lambda)$. Note that it follows from the decomposition of π into λ and σ that π is counted with $(-1)^{\ell(\pi)} a^{w(\pi)}$, where

$$w(\pi) = \sum_{i=1}^{\ell(\pi)} \lceil (\pi_i - \pi_{i+1})/2 \rceil = \ell(\lambda) + \ell(\sigma).$$

Let λ_e be the largest even part of λ . For convenience, we define $\lambda_e = 0$ if there is no even part in λ . We now compare λ_e and the largest part of σ , namely σ_1 .

Case 1: If $\sigma \neq \emptyset$ and $\lambda_e < \sigma_1$, then we move σ_1 to λ . We denote the resulting partitions by λ^* and σ^* . Since λ has even upper even parity index, the least part of λ is 1, which is also the least part of λ^* . Thus, the upper even parity index of λ^* is still even. Furthermore, since $\lambda_e < \sigma_1 \leq 2\ell(\lambda)$ and the parts of λ differ by at most 2, λ^* still has parts differing by at most 2.

Case 2: If $\lambda_e \neq 0$ and $\sigma_1 \leq \lambda_e$, then we move λ_e to σ . We denote the resulting partitions by λ^* and σ^* . Since λ has even upper even parity index, the least part of λ is 1, which is also the least part of λ^* . Thus, the upper even parity index of λ^* is even. Note that $\lambda_e < 2\ell(\lambda)$ since $\lambda_i - \lambda_{i+1} \leq 2$ and $\lambda_{\ell(\lambda)} = 1$. Since $\sigma_1 \leq \lambda_e < 2\ell(\lambda)$, the largest part of σ^* is less than or equal to $2\ell(\lambda) - 2$, which is equal to the number of parts of λ^* .

Let $\pi^* = \lambda^* + \sigma^*$. Then,

$$w(\pi) = \ell(\lambda) + \ell(\sigma) = \ell(\lambda^*) + \ell(\sigma^*) = w(\pi^*)$$

with $|\ell(\pi) - \ell(\pi^*)| = |\ell(\lambda) - \ell(\lambda^*)| = 1$. This shows that our mapping is a sign reversing involution under which $\pi = (2n-1, 2n-3, \dots, 3, 1)$ remains fixed. The generating function of such partitions is the right-hand side of (3.1). Therefore, the identity holds true. \square

4. UPPER PARITY INDICES IN PARTITIONS INTO DISTINCT PARTS

In this section, we prove Theorem 1.2 combinatorially. We first rewrite the theorem as follows.

$$D_o(\infty) = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^n y^{2k} q^{(n-k)^2+k^2+n+k}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q + \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^n y^{2k-1} q^{(n-k)^2+k^2+n-k}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k-1 \end{bmatrix}_q, \quad (4.1)$$

$$D_e(\infty) = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^n y^{2k} q^{(n-k)^2+k^2+k}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q + \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^n y^{2k+1} q^{(n-k)^2+k^2+3k+1}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q. \quad (4.2)$$

In the following theorem, we first show that the inner summation of the first double summations in (4.2) is the generating function of partitions into n distinct parts with even upper even parity index.

Theorem 4.1. *The generating function of partitions into n distinct parts with even upper even parity index is*

$$\sum_{k=0}^n \frac{q^{(n-k)^2+k^2+k}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q.$$

Proof. By Theorem 3.1, it suffices to show that

$$\sum_{k=0}^n q^{(n-k)^2+k^2+k} \begin{bmatrix} n \\ 2k \end{bmatrix}_q = (-q; q)_{n-1} q^{n(n+1)/2}, \quad (4.3)$$

which is equivalent to

$$\sum_{k=0}^n q^{(n-2k)(n-2k-1)/2} \begin{bmatrix} n \\ 2k \end{bmatrix}_q = (-q; q)_{n-1}.$$

This follows from Lemma 2.1. Indeed, one can fully show identity (4.3) in terms of partitions by rearranging the parts of partitions generated by one side to obtain the partitions generated by the other side. We, however, omit the details. \square

Similarly, we can show that the generating function of partitions into n distinct parts with odd upper odd parity index is

$$\sum_{k=0}^n \frac{q^{(n-k)^2+k^2+n-k}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k-1 \end{bmatrix}_q.$$

For even upper odd parity index (resp. odd upper even parity index), we first take a partition λ with even upper even parity index (resp. odd upper odd parity index). Then the smallest part of λ must be odd. We now add 1 to each of the parts of λ . Then the resulting partition has even upper odd parity index (resp. odd upper even parity index). By Theorem 4.1, we can show that the generating function of partitions into n distinct parts with even upper odd parity index is

$$\sum_{k=0}^n \frac{q^{(n-k)^2+k^2+n+k}}{(q^2; q^2)_n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q.$$

Similarly, we can show that the generating function of partitions into n distinct parts with odd upper even parity index is

$$\sum_{k=0}^n \frac{q^{(n-k)^2+k^2+3k+1}}{(q^2; q^2)_n} \left[\begin{matrix} n \\ 2k+1 \end{matrix} \right]_q.$$

5. REMARKS

The first three open questions of Andrews are settled by S. Kim and the author in [6]. The proof of Theorem 1.2 given in Section 4 is essentially the same as that of K. Kursungoz [7].

Question 5 of Andrews follows from Franklin's involution for the Euler pentagonal number theorem [2, pp. 10–11]. We state the question and sketch a solution. For a partition λ , we denote by $I_{LO}(\lambda)$ (resp. I_{LE}) the maximum length of weakly increasing subsequences of the parts whose terms alternate in parity starting with an odd (resp. even) part. Let $p_o(r, m, n)$ be the number of partitions of n into m distinct parts with $I_{LO} = r$. We define

$$P_o(y, x; q) = \sum_{r, m, n \geq 0} p_o(r, m, n) y^r x^m q^n.$$

Then we have the following theorem.

Theorem 5.1 (Andrews [3]).

$$P_o(y, x; q) = \sum_{n=0}^{\infty} \frac{x^n y^n q^{n(n+1)/2} (-q/y; q)_n}{(q^2; q^2)_n}.$$

Question 5. It follows from an old formula of Rogers that

$$P_o(-1, 1; q) = \sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}).$$

Prove combinatorially.

Note that $\lambda_1 \equiv I_{LO}(\lambda) \pmod{2}$ for a partition λ . Thus,

$$P_o(-1, 1; q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q; q)_n} = \sum_{n=0}^{\infty} q^{n(3n+1)/2} (1 - q^{2n+1}),$$

where the second equality follows from Franklin's involution.

Another combinatorial proof of Theorem 1.1 is given by B. C. Berndt, B. Kim, and the author in [5]. K. Alladi [1] has devised a completely different proof of Theorem 1.1 and has also provided a number-theoretic interpretation of Theorem 1.1 as a weighted partition theorem.

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