

THE FINITE HEINE TRANSFORMATION AND CONJUGATE DURFEE SQUARES

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ABSTRACT. We introduce the idea of a conjugate Durfee square and use it to answer a combinatorial question regarding a finite form of the Heine transformation posed by G. E. Andrews in a recent paper.

1. INTRODUCTION

In a recent publication [3], Andrews gave the following finite version of the Heine transformation:

Theorem 1.1 (Andrews). *For any n , we have*

$$\sum_{k=0}^n \frac{(q^{-n})_k (\alpha)_k (\beta)_k}{(q)_k (\gamma)_k (q^{1-n}/\tau)_k} q^k = \frac{(\beta, \alpha\tau)_n}{(\gamma, \tau)_n} \sum_k \frac{(q^{-n})_k (\gamma/\beta)_k (\tau)_k}{(q)_k (\alpha\tau)_k (q^{1-n}/\beta)_k} q^k. \quad (1.1)$$

In [3] Andrews asked for a combinatorial proof of Theorem 1.1 along the lines of his proof of Heine's ${}_2\phi_1$ transformation formula when n tends to infinity [1]. This paper provides such a proof.

2. CONJUGATE DURFEE SQUARES AND PRELIMINARY RESULTS

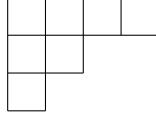
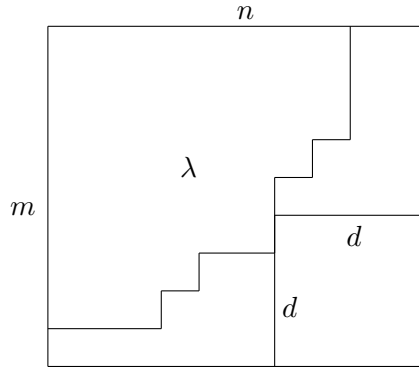
We define a *partition* of a positive integer n as a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 + \dots + \lambda_k = n$ with $\lambda_i \geq \lambda_{i+1}$. We relax our definition of a partition which may have 0 as a part if necessary. We refer to each λ_i as a part of our partition and denote by $|\lambda|$ the sum of its parts. We denote the number of parts of λ as $\ell(\lambda)$. For example, there are 7 partitions of 5, namely

$$(5), \quad (4, 1), \quad (3, 2), \quad (3, 1, 1), \quad (2, 2, 1), \quad (2, 1, 1, 1), \quad (1, 1, 1, 1, 1).$$

To each partition we can associate a Ferrers diagram. Each part of the partition is given as a row of boxes, each row aligned and put in descending order. Figure 1 represents the Ferrers diagram of $(4, 2, 1)$.

For a partition λ into at most m parts less than or equal to n , we define the (m, n) -conjugate Durfee square as the largest square that can fit with the Ferrers diagram of λ inside of a $m \times n$ rectangle without the two overlapping. Figure 2 illustrates the (m, n) -conjugate Durfee square. It is simple to see that for a given partition, the (m, n) -conjugate Durfee square is unique.

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FIGURE 1. The Ferrers diagram of $(4, 2, 1)$.FIGURE 2. The (m, n) -conjugate Durfee square with side d .

The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q)_n}{(q)_k (q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} (a)_0 &:= (a; q)_0 = 1, \\ (a)_n &:= (a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \geq 1. \end{aligned}$$

A partition theoretic interpretation of the q -binomial coefficient is as follows:

$$\begin{bmatrix} M + N \\ M \end{bmatrix} = \sum_{\lambda} q^{|\lambda|},$$

where the sum is over all partitions λ whose Ferrers diagram can fit inside an $M \times N$ rectangle.

For more information on partitions, Ferrers diagrams or the q -binomial coefficient, see [2].

We prove the following lemma combinatorially, which is well known in the literature.

Lemma 2.1.

$$\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n - k \\ j \end{bmatrix} = \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n - j \\ k \end{bmatrix}$$

Proof. Note that the q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ counts many interesting combinatorial objects including the partitions with Ferrers diagram fitting inside an $(n - k) \times k$ rectangle. Here,

we use inversions of permutations, namely,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{w \in \text{Per}(0^k, 1^{n-k})} q^{\text{inv}(w)},$$

where $\text{Per}(0^k, 1^{n-k})$ is the set of permutations of k 0's and $(n-k)$ 1's, and $\text{inv}(w)$ is the number of inversions in w . Adopting this interpretation, we see that

$$\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ j \end{bmatrix} = \sum_{w \in \text{Per}(0^k, 1^{n-k-j}, 2^j)} q^{\text{inv}(w)},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ accounts for the inversions between k 0's and $(n-k)$ 1 or 2's, and $\begin{bmatrix} n-k \\ j \end{bmatrix}$ accounts for the inversions between $(n-k-j)$ 1's and j 2's. By counting the inversions between 2's and 0 or 1's first, and then the inversions between 0's and 1's, we obtain

$$\begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix},$$

which completes the proof. \square

It should be noted that one can combinatorially interchange from the partition and the permutation interpretations of the q -binomial coefficient. Suppose we are given the partition $\lambda = (\lambda_1, \lambda_2, \dots)$ where $\lambda_1 \leq n-k$ and $l(\lambda) \leq k$. We can obtain the permutation of $(0^k, 1^{n-k})$ by first considering

$$\underbrace{(0 \cdots 0)}_k \underbrace{1 \cdots 1}_{n-k}. \quad (2.1)$$

We move the first 0 to the right past λ_1 1's, the next 0 to the right past λ_2 1's, and so on. It should be clear that $|\lambda| = \text{inv}(w)$. We can consider an example with $n = 8$, $k = 3$ and $\lambda = (4, 2, 1)$. The corresponding permutation is (10101101). More can be found on this correspondence in [2].

We review a bijection that was first introduced by the second author in [5] to establish a combinatorial proof for Ramanujan's ${}_1\psi_1$ summation formula. Recall the q -binomial theorem [4]:

$$\sum_{n=0}^{\infty} \frac{(-a; q)_n}{(q; q)_n} (zq)^n = \frac{(-azq; q)_{\infty}}{(zq; q)_{\infty}}. \quad (2.2)$$

Yee's bijection. For a positive integer n , let π be a partition into nonnegative distinct parts less than n and σ a partition into exactly n parts. We define μ by

$$\mu_i = \sigma_{n-\pi_i} + \pi_i, \quad \text{for all } 1 \leq i \leq \ell(\pi), \quad (2.3)$$

and let ν be the partition consisting of the remaining $n - \ell(\pi)$ parts of σ . Then, it can be easily seen that μ has distinct parts. It also follows from the construction that μ and ν are uniquely determined by π and σ . Thus, this map is reversible. The left-hand side of (2.2)

generates the pairs of (π, σ) and the right-hand side generates the pairs of (μ, ν) . The map is a bijection between the two sets of such pairs of partitions.

3. THE FINITE HEINE TRANSFORMATION

In this section, we will demonstrate a combinatorial proof of Theorem 1.1 along the lines of Andrews's proof of Heine's ${}_2\phi_1$ transformation formula. We start by proving a special case of Theorem 1.1. By replacing α, τ, γ by $-\alpha, \tau q, \gamma\beta$, respectively, and letting β approach 0 in (1.1), we obtain the following lemma.

Lemma 3.1. *We have*

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k = \frac{(-\alpha \tau q)_n}{(\tau q)_n}. \quad (3.1)$$

Proof. Let μ be a partition into distinct parts less than or equal to n and ν be a partition into parts less than or equal to n . Then the right-hand side of (3.1) generates such pairs of partitions, namely

$$\frac{(-\alpha \tau q)_n}{(\tau q)_n} = \sum_{\mu, \nu} \tau^{\ell(\mu) + \ell(\nu)} \alpha^{\ell(\mu)} q^{|\mu| + |\nu|}.$$

Let $m = \ell(\mu) + \ell(\nu)$. We apply the reverse map of Yee's bijection to μ and ν and denote the resulting partitions by π and σ , where π is a partition into $\ell(\mu)$ nonnegative distinct parts and σ is a partition into exactly m parts less than or equal to n . We find the $(m, n+1)$ -conjugate Durfee square of σ and denote its side as k . Figure 3 illustrates the conjugate Durfee square

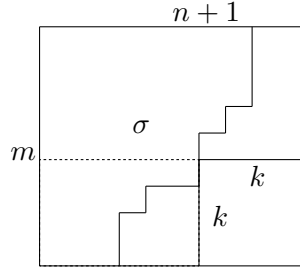


FIGURE 3. The $(m, n+1)$ -conjugate Durfee square of σ with side k .

of σ . Note that the k parts of σ below the dashed line are less than or equal to $n - k + 1$; the other parts above the dashed line are larger than or equal to $n - k + 1$, and less than or equal to n . Thus, the generating function of σ is

$$\sum_{\sigma} \tau^{\ell(\sigma)} q^{|\sigma|} = \begin{bmatrix} n \\ k \end{bmatrix} \frac{(\tau q)^k}{(\tau q^{n-k+1})_k}.$$

Furthermore, our process ensures that π has no part exceeding $k - 1$. Suppose that $\pi_1 \geq k$. Then, by Yee's bijection (2.3), we see that

$$\mu_1 = \sigma_{m-\pi_1} + \pi_1 \geq \sigma_{m-k} + k \geq n + 1 - k + k = n + 1,$$

which is a contradiction to the fact that μ has parts less than or equal to n . Thus, the generating function of π is $(-\alpha)_k$. Therefore, summing over all possible values of π and σ , we obtain

$$\sum_{k=0}^n \sum_{\pi, \sigma} \tau^{\ell(\sigma)} \alpha^{\ell(\pi)} q^{|\pi|+|\sigma|} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k,$$

which completes the proof. \square

We now prove Theorem 1.1 combinatorially. We first make some change of variables. Allowing $\alpha, \tau, \gamma \rightarrow -\alpha, \tau q, -\gamma\beta$ followed by $\beta \rightarrow \beta q$ in Theorem 1.1 yields the equivalent identity,

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k \frac{(-\gamma\beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\gamma)_k}{(\beta q^{n-k+1})_k} (\beta q)^k \frac{(-\alpha\tau q^{k+1})_{n-k}}{(\tau q^{k+1})_{n-k}}. \quad (3.2)$$

Theorem 3.2. *Equation (3.2) is valid.*

Proof. We start by interpreting the left hand side of (3.2). We will show that the left and right hand side of (3.2) generate 7-tuples of partitions. We first note that the term

$$\frac{(-\gamma\beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}}$$

on the left hand side of (3.2) can be interpreted as a strict partition μ with all parts exceeding k and no part exceeding n , and a partition ν with all parts exceeding k and no part exceeding n . As we did in the proof of Lemma 3.1, we apply the reverse map of Yee's bijection to μ and ν to obtain a pair of partitions π and σ , where π is a partition with nonnegative distinct parts and σ is a partition with all parts exceeding k and no part exceeding n . Let j be the side of the $(\ell(\sigma), n+1)$ -conjugate Durfee square of σ . Then, all the parts of π are less than j . Thus, using Lemma 3.1, we can see that

$$\begin{aligned} & \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k \frac{(-\gamma\beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}} \\ &= \sum_{k=0}^n \sum_{j=0}^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k \begin{bmatrix} n-k \\ j \end{bmatrix} \frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q^{k+1})^j. \end{aligned}$$

The interpretation is the same for the right hand side of (3.2), namely

$$\begin{aligned} & \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q)^j \frac{(-\alpha\tau q^{j+1})_{n-j}}{(\tau q^{j+1})_{n-j}} \\ &= \sum_{j=0}^n \sum_{k=0}^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q)^j \begin{bmatrix} n-j \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q^{j+1})^k. \end{aligned}$$

We can now see that the left hand side of (3.2) generates 7-tuples of partitions $(\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \lambda^6, \lambda^7)$, where $\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \lambda^6$ and λ^7 are generated by $\begin{bmatrix} n \\ k \end{bmatrix}, (\tau q)^k (\beta q^{k+1})^j, (-\alpha)_k, 1/(\tau q^{n-k+1})_k, \begin{bmatrix} n-k \\ j \end{bmatrix}, (-\gamma)_j, 1/(\beta q^{n-j+1})_j$, respectively; while the right hand side generates 7-tuples of partitions $(\mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \mu^7)$, where $\mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6$ and μ^7 are generated by $\begin{bmatrix} n \\ j \end{bmatrix}, (\beta q)^j (\tau q^{j+1})^k, (-\gamma)_j, 1/(\beta q^{n-j+1})_j, \begin{bmatrix} n-j \\ k \end{bmatrix}, (-\alpha)_k, 1/(\tau q^{n-k+1})_k$, respectively.

To show (3.2), given $\lambda^3, \lambda^4, \lambda^6$ and λ^7 , we take $\mu^3 = \lambda^6, \mu^4 = \lambda^7, \mu^6 = \lambda^3$ and $\mu^7 = \lambda^4$. To construct a bijection between (λ^1, λ^5) and (μ^1, μ^5) we apply Lemma 2.1, noting that we can combinatorially interchange between partitions and permutations as seen in the Section 2. Lastly, we must construct the bijection between λ^2 and μ^2 . We note that λ^2 is a partition with k 1's each marked with a τ and j $k+1$'s each marked with a β . We subtract k from each of the j parts of size $k+1$ and add j to each of the k parts of size 1. Thus, we have a partition with j 1's each marked with a β and k $j+1$'s each marked with a τ . It is easy to see this is μ^2 . \square

4. CONCLUSION

We see as $n \rightarrow \infty$ that our conjugate Durfee square gets pushed further to the right, eliminating all of the parts which lie above it and reducing our proof down to a proof similar to Andrews. In terms of Ferrers diagrams, the integral part of Andrews proof of the Heine transformation is removing a rectangle, flipping it on its diagonal and reinserting it. We can see this in our proof when we show the bijection from λ^2 to μ^2 .

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