Name Math 230 Quiz 7

1.(5 points) Find the local maximum and minimum values and saddle points of the function $f(x, y) = x^3 + xy^2 + 3x^2 + y^2$.
Solution:
We compute the partial derivatives f_x and f_y and set them equal to zero.

$$f_x = 3x^2 + y^2 + 6x = 0 \quad (1)$$
$$f_y = 2xy + 2y = 0 \quad (2)$$

Divide both side of (2) by 2 and factor it. we have

$$(x + 1)y = 0$$

So either $x = -1$ or $y = 0$.

If $x = -1$, plug in (1), we get $y^2 - 3 = 0$. It follows that we have critical points $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$.
If $y = 0$, plug in (1), we get $3x^2 + 6x = 0$, thus $x = 0$ or $x = -2$. The critical points are $(0, 0)$ and $(-2, 0)$.

Now we apply the second derivative test to see whether the points are rel. max/min or saddle points.

$$f_{xx} = 6x + 6 \qquad f_{xy} = 2y \qquad f_{yy} = 2x + 2 \qquad D = f_{xx}f_{yy} - f_{xy}^2$$

Plug in the points, we have the following table

	f_{xx}	D
$(-1, \sqrt{3})$	0	-12
$(-1, \sqrt{3})$	0	-12
(0,0)	6	12
(-2,0)	-6	12

Recall our second derivatives test says the following

- $D < 0 \Longrightarrow$ saddle point.
- $D > 0$ and $f_{xx} > 0 \Longrightarrow$ rel. minimum.
- $D > 0$ and $f_{xx} \leq 0 \Longrightarrow$ rel. maximum.

It is clear that both $(0, \sqrt{3})$ and $(0, -\sqrt{3})$ are saddle points. $f(x, y)$ has a local minimum at $(0, 0)$ with minimum value $f(0, 0) = 0$, also a local maximum at $(-2, 0)$ with maximum value $f(-2, 0) = 4$.

2.(5 points))Use Lagrange multiplier to find the maximum and minimum of the function $f(x, y, z) = \frac{1}{\sqrt{2}}x + yz$ under the constraint $x^2 + y^2 + z^2 = 1$.

Solution:
we set up the equation system to find the critical points first.

$$\frac{1}{\sqrt{2}} = 2x\lambda \tag{3}$$
$$z = 2y\lambda \tag{4}$$
$$y = 2z\lambda \tag{5}$$
$$x^2 + y^2 + z^2 = 1 \tag{6}$$

Multiply (4) and (5) together, we have $yz = 4yz\lambda^2$. Notice this is the same as $(1-4\lambda^2)yz = 0$. So we have four possible situations: $y = 0$, $z = 0$, $\lambda = 1/2$ or $\lambda = -1/2$.

Notice that from (4) and (5), $y = 0$ would imply $z = 0$ and vice versa. Plugging in (6), we get $x = \pm 1$, so the only critical points with y or z equals 0 are $(1, 0, 0)$ and $(-1, 0, 0)$.

Now if $\lambda = 1/2$, from (3), we get $x = \frac{1}{\sqrt{2}}$. Also it follows from either (4) or (5) that $y = z$. Plug all the above information into (6), we get $x = 1/\sqrt{2}, y = z = \pm 1/2$, i.e., the critical points in this situation are $(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2})$.

Similarly, if $\lambda = -\frac{1}{2}$, we have critical points $(-\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2})$.

In summary, we have exact 6 critical points. And at these points, the value of $f(x, y, z)$ are

Point	$f(x,y,z)$
$(1, 0, 0)$	$\frac{1}{\sqrt{2}}$
$(-1, 0, 0)$	$-\frac{1}{\sqrt{2}}$
$(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$	$\frac{3}{4}$
$(\frac{1}{\sqrt{2}}, -\frac{1}{2}, -\frac{1}{2})$	$\frac{3}{4}$
$(-\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2})$	$-\frac{3}{4}$
$(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2})$	$-\frac{3}{4}$

Since $\frac{3}{4} > \frac{1}{\sqrt{2}}$, we conclude that our function has maximum value $\frac{3}{4}$, and minimum $-\frac{3}{4}$.