Solutions to Quiz 7

(5 points) 1. Let $f(x) = 1 - x^2, \quad 0 < x < 2.$

(1) Find the odd periodic extension (of period T=4) of f(x), sketch the graph of the extended, periodic function, on the interval $(-2, 6)$

Solution :

$$\begin{aligned} f_{odd}(x) &= 1 - x^2, \quad 0 < x < 2 \\ &= -1 + x^2, \quad -2 < x < 0 \end{aligned}$$

The graph has been drawn in class on the blackboard!

$$\begin{aligned} f_{per}(x) &= 1 - x^2, \qquad -4 < x < -2 \\ &= -1 + x^2, \qquad -2 < x < 0 \\ &= 1 - x^2, \qquad 0 < x < 2 \\ &= -1 + x^2, \qquad 2 < x < 4 \\ &= 1 - x^2, \qquad 4 < x < 6 \end{aligned}$$

We will use the periodic function $f_{per}(x)$ in the following question.

(2) To what values would the Fourier series converge at x=0, x=-2, and x=5?

Solution :

At x=0, Fourier series converge to

$$\frac{f_{per}(0^-) + f_{per}(0^+)}{2} = \frac{-1+1}{2} = 0.$$

At x=-2, Fourier series converge to

$$\frac{f_{per}(-2^-) + f_{per}(-2^+)}{2} = \frac{-3+3}{2} = 0.$$

At x=5, $f_{per}(x)$ is continuous, so Fourier series converge to

$$f_{per}(5) = f(1) = 0.$$

(5 points) 2. (1)Solve the non-homogeneous boundary value problem,

$$\begin{aligned} u_t &= 4u_{xx}, \qquad 0<x<20, \quad t>0 && (1)\\ u(0,t) &= 30, \qquad u(20,t)=20, \qquad t>0 && (2)\\ u(x,0) &= 10-x, \qquad 0<x<20. && (3) \end{aligned}$$

Solution :

We are considering $u(x,t) = v(x) + w(x,t)$, the steady-state solution v(x) satisfies

$$\begin{aligned} v''(x) &= 0, \qquad 0<x<20 && (4)\\ v(0) &= 30, \qquad v(20)=20. && (5) \end{aligned}$$

Integrating both sides of (4) twice, we know

$$v(x) = Ax + B,$$

Using boundary conditions (5),

$$\begin{aligned} &30 = v(0) = A\cdot 0 + B = B \qquad \Rightarrow B = 30,\\ &20 = v(20) = 20A + B = 20A + 30 \qquad \Rightarrow A = -\frac{1}{2},\\ &\Rightarrow v(x) = -\frac{1}{2}x + 30. \end{aligned}$$

Therefore, the transient solution $w(x,t)$ would satisfy

$$\begin{aligned} w_t &= 4w_{xx}, \qquad 0<x<20, \quad t>0 && (6)\\ w(0,t) &= w(20,t) = 0, \qquad t>0 && (7)\\ w(x,0) &= 10 - x - v(x) = -\frac{x}{2} - 20, \qquad 0<x<20. && (8) \end{aligned}$$

Using separation of variables, we could get a family of

$$w_n(x,t) = sin(\frac{n\pi x}{20})\cdot e^{-4(\frac{n\pi}{20})^2 t}$$

satisfying (6) and (7). To satisfy the initial condition (8), we are considering

$$w(x,t) = \sum_{n=1}^{+\infty} A_n\, w_n(x,t) = \sum_{n=1}^{+\infty} A_n\, sin(\frac{n\pi x}{20})\cdot e^{-4(\frac{n\pi}{20})^2 t}.$$

which also satisfies the homogeneous BVP (6) and (7). Moreover, (8) tells us

$$-\frac{x}{2} - 20 = w(x,0) = \sum_{n=1}^{+\infty} A_n\, sin(\frac{n\pi x}{20}).$$

Using Euler-Fourier Formula for sine series, we could get all coefficients A_n

$u(x,t) = v(x) + w(x,t) = -\frac{1}{2}x + 30 + \sum_{n=1}^{+\infty} A_n\, sin(\frac{n\pi x}{20}) \cdot e^{-4(\frac{n\pi}{20})^2 t}$

(2) Find $lim_{t\to\infty} u(1,t)$.

Solution :

$$lim_{t\to\infty} u(1,t) = lim_{t\to\infty} v(1) + lim_{t\to\infty} w(1,t) = v(1) = \frac{59}{2}$$

The reason is that, for each term, $w_n(x,t) = A_n\, sin(\frac{n\pi x}{20}) \cdot e^{-4(\frac{n\pi}{20})^2 t}$ in $w(x,t)$, it is an exponential decay!