Math 251 Quiz 5 Answers

1. Find solution to the two-point boundary value problem

$$y'' + \lambda y = 0, \qquad y'(0) = y'(3) = 0$$

Answer Let us discuss in three cases

Case (1) $\lambda = 0$

the equation becomes $y'' = 0$, integrate once, we get $y'(x) \equiv C$, using the condition

$$y'(0) = y'(3) = 0,$$

$\Longrightarrow$C=0. $\Longrightarrow$ $y(x) \equiv C$ is the particular solution, where C is any constant.

Case (2) $\lambda > 0$

The characteristic equation is $r^2 - \lambda = 0$, $\Longrightarrow r = \pm\sqrt{\lambda}i$, so the general solution is

$$\begin{aligned} y(x) &= C_1 cos(\sqrt{\lambda}x) + C_2 sin(\sqrt{\lambda}x) \\ y'(x) &= -\sqrt{\lambda}C_1 sin(\sqrt{\lambda}x) + \sqrt{\lambda}C_2 cos(\sqrt{\lambda}x) \\ 0 &= y'(0) = \sqrt{\lambda}C_2 \Longrightarrow C_2 = 0; \\ 0 &= y'(3) = -\sqrt{\lambda}C_1 sin(3\sqrt{\lambda}) \end{aligned}$$

If $C_1 = 0$, then we only get the solution $y(x) \equiv 0$.

Otherwise, $\sin(3\sqrt{\lambda}) = 0$,

$$\begin{aligned} \Longrightarrow 3\sqrt{\lambda} &= n\pi, \qquad n = 1,\, 2,\, 3\cdots \text{(n doesn't start from 0, since n=0, then } \lambda = 0) \\ \lambda &= (\frac{n\pi}{3})^2, \qquad n = 1,\, 2,\, 3\cdots \end{aligned}$$

Now we are using the subscript λ_n for each n.

$\lambda_n = (\frac{n\pi}{3})^2$ are the eigenvalues.

For each λ_n, the particular solution, denoted by $y_n(x)$ (second subscript!), equals

$$y_n(x) = A_n cos(\sqrt{\lambda_n}x) = A_n cos(\frac{n\pi x}{3}) \qquad n = 1,\, 2,\, 3\cdots$$

Case (3) $\lambda < 0$

Denote $-\lambda = (\sqrt{-\lambda})^2 = \alpha^2$, here $\alpha > 0$.

The characteristic equation is $r^2 - \alpha^2 = 0$, $\Longrightarrow r = \pm\alpha$, so the general solution is

$$\begin{aligned} y(x) &= C_1 e^{\alpha x} + C_2 e^{-\alpha x}, \\ y'(x) &= \alpha C_1 e^{\alpha x} - \alpha C_2 e^{-\alpha x}, \\ 0 &= y'(0) = \alpha C_1 - \alpha C_2 \Longrightarrow C_1 = C_2, \\ 0 &= y'(3) = \alpha C_1 e^{3\alpha} - \alpha C_2 e^{-3\alpha} = \alpha C_1(e^{3\alpha} - e^{-3\alpha}) \end{aligned}$$

e^x is an increasing function, so $e^{3\alpha} - e^{-3\alpha} > 0 \Longrightarrow C_1 = C_2 = 0$. y(x)=0

Combining all three cases, all solutions are

$$y_n(x) = A_n cos(\frac{n\pi x}{3}), \qquad n = 0,\, 1,\, 2,\, 3\cdots$$

2. Find the solution to the heat conduction problem

$$\begin{aligned} 5u_{xx} &= u_t \qquad \text{for} \quad 0 < x < 3\,, \qquad t > 0 \\ u'(0,t) &= u'(3,t) \;=\; 0 \qquad \text{for} \quad t > 0 \\ u(x,0) &= 10 + 4\sin\frac{2\pi x}{3} - 2\sin\frac{4\pi x}{3} \qquad \text{for} \quad 0 \le x \le \pi \end{aligned}$$

You can use the result about two-point boundary value problem from problem *1* directly.

Answer Using separation of variables, we assume

$$u(x,t) = X(x)T(t).$$

Plugging into $u_t = 5u_{xx}$, $\Longrightarrow$

$$X(x)T'(t) = 5X''(x)T(t)$$

Divide both sides by $5X(x)T(t)$, $\Longrightarrow$

$$\frac{T'(t)}{5T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad (constant)$$

We get two linear homogeneous ODEs

$$T'(t) + 5\lambda T(t) = 0 \tag{1}$$
$$and \; X''(x) + \lambda X(x) = 0 \tag{2}$$
$$X'(0) = X'(3) = 0 \tag{3}$$

For equations (2) and (3), using the results in the first problem, for each n,

$$\lambda_n = (\frac{n\pi}{3})^2, \qquad X_n(x) = A_n cos(\frac{n\pi x}{3}) \qquad n = 0,\, 1,\, 2,\, 3\cdots$$

This suggests us for each n, we have a solution

$$u_n(x,t) = X_n(x)T_n(t)$$
$$\frac{T_n'(t)}{5T_n(t)} = \frac{X_n''(x)}{X_n(x)} = -\lambda_n = -(\frac{n\pi}{3})^2 \tag{4}$$

The first equality in (4) provides us with

$$T'(t) + 5(\frac{n\pi}{3})^2 T(t) = 0 \qquad \Longrightarrow T(t) = e^{-5(\frac{n\pi}{3})^2 t}$$

Thus we get a family of solutions

$$u_n(x,t) = A_n cos(\frac{n\pi x}{3})e^{-5(\frac{n\pi}{3})^2 t} \qquad n = 0,\, 1,\, 2,\, 3\cdots$$

Where each $u_n(x,t)$ satisfies (D.E.) and (B.C). Now we need to take (I.C.) into account. Considering the finite sum

$$\sum_{n=0}^{N} u_n(x,t) = \sum_{n=0}^{N} A_n cos(\frac{n\pi x}{3})e^{-5(\frac{n\pi}{3})^2 t},$$

(I.C) tells us

$$\sum_{n=0}^{N} A_n cos(\frac{n\pi x}{3}) = 10 + 4\sin\frac{2\pi x}{3} - 2\sin\frac{4\pi x}{3}$$

By comparing the coefficients of all these trig-polynomials, we conclude

$$A_0 = 10,\ A_2 = 4,\ A_4 = -2$$

In sum, particular solution is

$$10 + 4cos(\frac{2\pi x}{3})e^{-5(\frac{2\pi}{3})^2 t} - 2cos(\frac{4\pi x}{3})e^{-5(\frac{4\pi}{3})^2 t}$$