1. Let \( f : X \times [a, b] \to \mathbb{R} \) where \([a, b]\) is a finite interval. Assume that the function \( x \mapsto f(x, y) \) is integrable for every \( y \in [a, b] \). Assume also that \( \partial_y f(x, y) \) exists for every \( (x, y) \) and there exists a function \( g \in L^1 \) such that \( |\partial_y f(x, y)| \leq g(x) \) for all \( (x, y) \). Show that the function \( x \mapsto \partial_y f(x, y) \) is measurable for every \( y \), and

\[
\partial_y \int f(x, y) d\mu(x) = \int \partial_y f(x, y) d\mu(x).
\]

**Solution:** Fix \( y \in [a, b] \) and let \( y_n \in (a, b) \) be any sequence converging to \( y \). Define

\[
f_n(x) = \frac{1}{y_n - y} \left[ f(x, y_n) - f(x, y) \right].
\]

The right-hand side defines a measurable function in \( x \). Since \( f_n(x) \to \partial_y f(x, y) \), it follows that \( x \mapsto \partial_y f(x, y) \) is a measurable function on \( X \). Applying the mean-value theorem we get

\[
|f_n(x)| \leq \sup_{y \in [a, b]} |\partial_y f(x, y)| \leq g(x).
\]

By DCT,

\[
\partial_y \int f(x, y) d\mu(x) = \lim_{n \to \infty} \int_X f_n(x) d\mu(x) = \int_X \lim_{n \to \infty} f_n(x) d\mu(x) = \int \partial_y f(x, y) d\mu(x).
\]

2. Prove:

(a) \( \|f_n - f\|_\infty \to 0 \) if and only if there exists \( E \in \mathcal{M} \) such that \( \mu(E^c) = 0 \) and \( f_n \to f \) uniformly on \( E \).

(b) \((L^\infty, \| \cdot \|_\infty)\) is a Banach space.

**Solution** (a) Assume that \( f_n \to f \) uniformly on \( E \) and that \( \mu(E^c) = 0 \). Given \( \varepsilon > 0 \), there is \( N \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( x \in E \) and all \( n \geq N \). Then \( \|f_n - f\|_\infty \to 0 \) as \( n \to \infty \). Conversely, let \( E_n^c = \{ x \mid |f_n(x) - f(x)| < 1/n \} \). Since \( |f_n - f|_\infty \to 0 \), for every \( n \in \mathbb{N} \) there exists \( N_n \in \mathbb{N} \) such that \( \mu(E_n^c) = 0 \) for all \( k \geq N_n \). Set \( E^n = \bigcup_{k \geq N_n} E_k \) and \( E = \bigcap_{n \geq 1} E^n \). Then \( \mu(E) = 0 \). Take \( \varepsilon > 0 \) and let \( N = N_n \) such that \( 1/n < \varepsilon \). If \( x \in E \), then \( x \in E_n \) so that \( |f_k(x) - f(x)| < 1/n < \varepsilon \) for all \( k \geq N = N_n \) showing that \( f_n \to f \) uniformly on \( E \).

(b) Assume that \( \|f_n - f\|_m \to 0 \) as \( m \to \infty \). Arguing as above, there exists a \( \mu \)-measurable set \( E \) with \( \mu(E^c) = 0 \) and such that the complex valued sequence \( (f_n) \) is uniformly Cauchy on \( E \). Hence for every \( x \in E \), there is \( f(x) \in \mathbb{C} \) such that \( f_n(x) \to f(x) \). Set \( f(x) = 0 \) if \( x \in E^c \). Then \( f \) is measurable on \( X \) and \( f_n \to f \) uniformly on \( E \), that is, \( \|f_n - f\|_\infty \to 0 \). Moreover, there is \( N \in \mathbb{N} \) such that \( |f(x) - f_N(x)| \leq 1 \) for all \( x \in E \). So, \( |f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + \|f_N\|_\infty \) for all \( x \in E \). Since \( \mu(E^c) = 0 \), \( \|f\|_\infty \leq 1 + \|f\|_\infty < \infty \). Hence \( f \in L^\infty \).

3. Assume that \( \mu(X) = \infty \) and \( X = \bigcup_{k \geq 1} E_k \) with \( \mu(E_k) < \infty \).

(a) Show that there are sets \( F_k \in \mathcal{M} \) such that \( X = \bigcup_{k \geq 1} F_k \) and \( 1 \leq \mu(F_k) < \infty \) for every \( k \).

(b) Use this fact to construct a function \( f \in L^p \) for all \( 1 < p \leq \infty \) but \( f \not\in L^1 \).

**Solution:** (a) We may assume that the original sets \( E_k \) are disjoint. (Otherwise,
consider \( E'_k = E_1 \) and \( E'_k = E_k \setminus \bigcup_{j \geq 1} E_j \). Then the sets \( E'_k \) are disjoint, \( X = \bigcup_k E_k = \bigcup_k E'_k \), and \( \mu(E'_k) \leq \mu(E_k) < \infty \). Since \( \sum_{k=1}^{\infty} \mu(E_k) = \mu(X) = \infty \), one finds a strictly increasing sequence \( (N_k)_{k \geq 0} \) \( N_0 = 1 \) and \( 1 \leq \sum_{j=N_k+1}^{N_{k+1}} \mu(E_j) = \mu(\bigcup_{j=N_k+1}^{N_{k+1}} E_j) \). Now set \( F_k = \bigcup_{j=N_k}^{N_{k+1}} E_j \) for \( k \geq 1 \).

(b) Let \( (F_k) \) be as above. Define \( f = \sum_{k=1}^{\infty} \frac{1}{k!} \chi_{F_k} \). Since \( F_k \) are disjoint, \( \int f^p = \sum_{k=1}^{\infty} \frac{1}{k! k^p} \mu(F_k) = \sum_{k=1}^{\infty} \mu(F_k) \mu(F_k) / k^p \leq \sum_{k=1}^{\infty} 1 / k^p < \infty \) if \( p > 1 \) (here we used that \( 1 / \mu(F_k) < 1 \) and \( p \geq 1 \)). So, \( f \in L^p \) for \( 1 < p \leq \infty \). If \( p = 1 \), then \( f = \sum_{k=1}^{\infty} (1 / k) = \infty \) showing that \( f \notin L^1 \).

4. Assume that \((f_n),(g_n)\) and \((h_n)\) are sequences of measurable functions satisfying \( g_n \leq f_n \leq h_n \) \( \mu\)-a.e. and \( g_n \to g, f_n \to f \) and \( h_n \to h \). Assume that \( g,h \in L^1 \) and

\[
\int g_n \to \int g \quad \text{and} \quad \int h_n \to \int h.
\]

By reworking the proof of the Lebesgue dominated convergence theorem, show that \( f \in L^1 \) and

\[
\int f_n \to \int f.
\]

**Solution:** By the assumptions, \( f_n - g_n \geq 0 \) and \( h_n - f_n \geq 0 \). The Fatou’s lemma applied to \( f_n - g_n \) gives

\[
\int f - \int g = \int (f - g) = \int \liminf_{n \to \infty} (f_n - g_n) \leq \liminf_{n \to \infty} \int (f_n - g_n) \leq \liminf_{n \to \infty} \int f_n - \int g.
\]

Similarly,

\[
\int h - \int f = \int (h - f) = \int \liminf_{n \to \infty} (h_n - f_n) \leq \liminf_{n \to \infty} \int (h_n - f_n) \leq \liminf_{n \to \infty} \left( \int h_n - \int f_n \right) = \int h - \limsup_{n \to \infty} \int f_n.
\]

Since \( g,h \in L^1 \), \( \liminf_{n \to \infty} \int f_n \geq \int f \geq \limsup_{n \to \infty} \int f_n \), and the result follows.

5. Let \( 1 \leq p < \infty \) and let \((f_n) \subset L^p \). Assume that \( f_n \to f \) \( \mu\)-a.e. with \( f \in L^p \). Use Problem 4 to show that

\[
\|f_n - f\|_p \to 0 \quad \text{if and only if} \quad \|f_n\|_p \to \|f\|_p.
\]

**Solution:** Since \( \|f\|_p \leq \|f_n\|_p + \|f_n - f\|_p \) and \( \|f_n\|_p \leq \|f\|_p + \|f_n - f\|_p \), it follows that \( \|f_n\|_p - \|f\|_p \leq \|f_n - f\|_p \). Hence if \( \|f_n - f\|_p \to 0 \), then \( \|f_n\|_p \to \|f\|_p \).

Conversely, assume that \( f_n \to f \) \( \mu\)-a.e. with \( f \in L^p \), and \( \|f_n\|_p \to \|f\|_p \). Then

\[
0 \leq |f_n - f|^p \leq 2^p \|f_n\|^p + |f|^p.
\]

Setting \( g_n = 0 \), \( h_n = 2^p \|f_n\|^p + |f|^p \), one gets \( h_n \to h = 2^p |f|^p \), \( f \to f \) (since \( \|f_n\|_p \to \|f\|_p \)). Since \( |f_n - f|^p \to 0 \), the problem 4 implies \( \|f_n - f\|^p \to 0 \), that is, \( \|f_n - f\|_p \to 0 \).