1. **The Cantor Set.** Denote by $C_0 = [0,1]$. Let $C_1$ be the set obtained by removing the middle third open interval of $C_0$, that is, $C_1 = [0,1/3] \cup [2/3,1]$. Next, let $C_2$ be the set obtained by removing the middle third open interval from each interval of $C_1$. Continuing this way, we obtain a decreasing sequence of closed sets $C_0 \supset C_1 \supset C_2 \supset \cdots$. The Cantor set $C$ is the intersection of all $C_k$’s, $C = \bigcap_{k=0}^{\infty} C_k$. Show:

(a) $C$ is totally disconnected, that is, if $x, y \in C$ and $x < y$, then there is $z \in (x, y)$ such that $z \notin C$.

(b) $C$ is perfect, that is, it is closed and without isolated points.

(c) $C$ is measurable and $m(C) = 0$.

**Solution:** (a) Let $x, y \in C$ and $x < y$. Assume that the interval $(x, y) \subset C$. Then $(x, y) \subset C_k$, $k \geq 0$. So, the length of $(x, y)$ is less or equal to $1/3^k$ for every $k$, contradiction.

(b) Take $x \in C$. Then $x$ belongs to the unique closed interval $I_k$ constituting $C_k$. The length of $I_k$ is equal to $1/3^k$. If $a_k$ is the left end point of $I_k$, then $a_k \in C$ and $|x - a_k| \leq 1/3^k$. So, the point $x$ is not isolated and the set $C$ is perfect.

(c) The set $C_k$ consists of $2^k$ disjoint closed intervals each of length $1/3^k$. Hence $m^*(C_k) \leq (2/3)^k$ and $m^*(C) = m^*(\bigcap_{k=0}^{\infty} C_k) \leq m^*(C_k) \leq (2/3)^k$ for every $k$. Hence $m^*(C) = 0$ showing that $C$ is measurable with $m(C) = 0$.

2. Similarly as the Cantor set, we construct a subset $C_\delta$ of $[0,1]$, except that at the $k$th step, each removed open interval has length $\delta 3^{-k}$, where $0 < \delta < 1$. Show that $C_\delta$ is measurable and $m(C_\delta) = 1 - \delta$.

**Solution:** Denote by $C_k$ the set obtained in the $k$th step. The set $C_\delta$ is equal $C_\delta = \bigcap_{k \geq 0} C_k$. Every $C_k$ is closed as a finite union of closed intervals. So, $C_\delta$ is closed as a intersection $C$ of closed sets. This implies that $C_\delta$ measurable. To calculate its measure, note that $C_k$ consists of $2^k$ disjoint closed intervals. To obtain $C_{k+1}$ one removes the middle interval of length $\delta/3^{k+1}$ from each of the intervals of $C_k$. Hence the measure of the removed intervals is equal to $2^k/3^{k+1}$. This shows that $m(C_{k+1}) = m(C_k) - \delta 2^k/3^{k+1} = \cdots = 1 - \delta \sum_{j=0}^{k} 2^j/3^{j+1}$. Since $C_k \setminus C_\delta$, using the continuity from above one finds

$$m(C_\delta) = \lim_k m(C_k) = \lim_k [1 - \delta \sum_{j=0}^{k} 2^j/3^{j+1}] = 1 - \delta.$$ 

3. Let $\delta = (\delta_1, \ldots, \delta_d)$ be a $d$-tuple of positive numbers $\delta_i$, and $E \subset \mathbb{R}^d$. Define $\delta E = \{(\delta_1 x_1, \ldots, \delta_d x_d) | (x_1, \ldots, x_n) \in E\}$. Prove that $\delta E$ is measurable whenever $E$ is measurable and that $m(\delta E) = \delta_1 \cdots \delta_d \cdot m(E)$. 

**MATH 501 - FALL 2007**

**Solutions to HOMEWORK 1**
Solution: Abbreviate \( r=\delta_1\cdots\delta_d \). If \( Q \) is a closed cube, then \( m_*(\delta Q) = |\delta Q| = r|Q| \). Moreover, if \( O \subseteq \mathbb{R}^d \) is open, then \( \delta O \) is also open. Let \( E \subseteq \mathbb{R}^d \) be measurable. Given \( \varepsilon > 0 \), there exists an open set \( O \) such that \( E \subseteq O \) and \( m_*(O \setminus E) \leq \varepsilon/(2r) \). By the definition of \( m_* \), there exists a sequence \( (Q_j)_{j \geq 1} \) of closed cubes such that \( O \setminus E \subseteq \bigcup_{j=1}^\infty Q_j \) and \( \sum_{j=1}^\infty |Q_j| \leq m_*(O \setminus E) + \varepsilon/(2r) \). Hence \( \sum_{j=1}^\infty |Q_j| \leq \varepsilon/r \). Since \( \delta E \subseteq \delta O \) and \( \delta O \setminus E \subseteq \bigcup_{j} \delta Q_j \), the countable sub-additivity of \( m_* \) implies
\[
m_*((\delta E \setminus E) \subseteq \sum_{j=1}^\infty |\delta Q_j| = r \sum_{j=1}^\infty |Q_j| < \varepsilon.
\]
Hence \( \delta E \) is measurable. To see that \( m_*(\delta E) = r \cdot m(E) \), let \( \varepsilon > 0 \) and let \( (Q_j) \) be a sequence of closed cubes such that \( E \subseteq \bigcup_{j} Q_j \) and \( \sum_{j=1}^\infty |Q_j| \leq m_*(E) + \varepsilon/r \).

Then \( \delta E \subseteq \bigcup_{j} \delta Q_j \). By countable sub-additivity of \( m_* \) we find that
\[
m_*((\delta E \setminus E) \subseteq \sum_{j=1}^\infty m_*(\delta Q_j) = r \sum_{j=1}^\infty |Q_j| \leq rm_*(E) + \varepsilon.
\]
This implies that \( m_*((\delta E \setminus E) \subseteq rm_*(E) \). Denote by \( 1/\delta = (1/\delta_1,\ldots,1/\delta_d) \) and note that \( (1/\delta)(\delta E) = E \). So,
\[
m_*(E) = m_*(1/\delta)(\delta E)) \leq \frac{1}{r} m_*(\delta E).
\]
Hence \( rm_*(E) \leq m_*(\delta E) \).

4. Assume that \( A \subseteq \mathbb{R}^p \) and \( B \subseteq \mathbb{R}^q \) are measurable. Prove that \( A \times B \) is a measurable subset of \( \mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q \) and that \( m(A \times B) = m(A)m(B) \). (Interpret \( 0 \cdot \infty \) as 0).

Solution:
(a) If \( A \subseteq \mathbb{R}^p \) and \( B \subseteq \mathbb{R}^q \) are closed cubes, then \( A \times B \) is a closed rectangle in \( \mathbb{R}^{p+q} \) and \( |A \times B| = |A| \cdot |B| \). Since the outer measure of a cube (of a rectangle) is equal to its volume, one concludes \( m(A \times B) = m(A)m(B) \).

(b) If \( A \subseteq \mathbb{R}^p \) and \( B \subseteq \mathbb{R}^q \) are open, then \( A \times B \) is open in \( \mathbb{R}^{p+q} \). Then \( A = \bigcup_{j \geq 1} Q_j \) and \( B = \bigcup_{k \geq 1} P_k \) where \( Q_j \)'s and \( P_k \)'s are almost disjoint closed cubes in \( \mathbb{R}^p \) and \( \mathbb{R}^q \), respectively. Moreover, \( m(A) = \sum_{j \geq 1} m(Q_j) \) and \( m(B) = \sum_{k \geq 1} m(P_k) \).

Since \( A \times B = \bigcup_{j,k \geq 1} Q_j \times P_k \), the rectangles \( Q_j \times P_k \) are almost disjoint, one concludes from 1. that
\[
m(A \times B) = \sum_{j,k \geq 1} m(Q_j \times P_k) = \sum_{j,k \geq 1} m(Q_j)m(P_k) = \sum_{j \geq 1} m(Q_j) \sum_{k \geq 1} m(P_k) = m(A)m(B).
\]

(c) Let \( A \) and \( B \) be bounded \( G_\delta \) sets. Then \( A = \bigcap_{j \geq 1} G_j \), \( B = \bigcap_{k \geq 1} H_k \) where \( G_j \) and \( H_k \) are open in \( \mathbb{R}^p \) and \( \mathbb{R}^q \), respectively. Since \( A \) is bounded, \( A \subseteq Q \) where \( Q \) is an open cube. Then \( A = \bigcup_{j \geq 1} [G_j \cap Q] \) and \( G_j \cap Q \) are open and bounded. Hence we may assume that the sets \( G_j \) and \( H_k \) are open and bounded. Considering decreasing sequence of open sets \( G_1 \cap G_2 \cap \cdots \cap G_j \) for \( j \geq 1 \), we
have \( A = \bigcap_{j=1}^{\infty} [G_1 \cap \cdots \cap G_j] \). Hence we may assume \( G_j \setminus A, H_j \setminus B \). Then 
\[ G_j \times H_j \setminus A \times B = \bigcap_{j=1}^{\infty} G_j \times H_j. \] By the continuity from above and by (a),
\[ m(A \times B) = \lim_{j} m(G_j \times H_j) = \lim_{j} m(G_j) = \lim_{j} m(H_j) = m(A) \times m(B). \]

(d) Let \( A \) be \( G_{\delta} \) sets and \( B \) be a bounded \( G_{\delta} \) sets. Then \( A = \bigcup_{j=1}^{\infty} A_j \) where \( A_j = A \cap B_j(0) \). Each \( A_j \) is a bounded \( G_{\delta} \) set and \( A_j \not\subset A \). Then \( A_j \times B \) is a bounded \( G_{\delta} \) and \( A \times B \). Using the continuity from below and (c), one finds 
\[ m(A) = \lim_{j} m(A_j) = \lim_{j} m(A_j \times B) = m(A \times B). \]

In the general case when both \( A \) and \( B \) are both \( G_{\delta} \) sets but applied to \( B \) shows that \( A \times B \) is measurable and 
\[ m(A \times B) = m(A)m(B). \]

(e) Let \( A \) be bounded and \( m(B) = 0 \). Then \( A \) is a subset of a closed cube \( P \subset \mathbb{R}^p \) and, given \( \varepsilon > 0 \), there is a sequence \( (Q_j) \) of closed cubes in \( \mathbb{R}^q \) such that \( B \subset \bigcup_{j} Q_j \) and \( \sum_{j} |Q_j| < \varepsilon |P| \). Then 
\[ A \times B \subset \bigcup_{j} P \times Q_j \] and \( \sum_{j} |P \times Q_j| < \varepsilon \).

Since \( \varepsilon > 0 \) is arbitrary, \( A \times B \) has measure 0 in \( \mathbb{R}^p \times \mathbb{R}^q \) and so, it is measurable.

If \( A \) is an arbitrary set in \( \mathbb{R}^p \), then \( A = \bigcup_{j} A_j \) where \( A_j = A \cap B_j(0) \). The set \( A_j \) is bounded, hence \( A_j \times B \) is of measure 0 in \( \mathbb{R}^p \times \mathbb{R}^q \).

(f) Finally, let \( A, B \) be measurable in \( \mathbb{R}^p \) and \( \mathbb{R}^q \), respectively. Then \( A = G \setminus N \) and \( H \setminus M \) where \( G, H \) are \( G_{\delta} \) sets such that \( A \subset G, B \subset H \), and \( N, M \) are sets of measure 0. Then 
\[ m(A) = m(G) \text{ and } m(B) = m(H), \]
and 
\[ m(A \times B) \leq m(G \times H) \] and from (1) 
\[ m(G \times H) \leq m(A \times B) \] Hence 
\[ m(A \times B) = m(G \times H) \] and 
\[ m(A \times B) = m(G \times H) = m(G)m(H) = m(A)m(B). \]

Here is much shorter proof of measurability of \( A \times B \) provided that \( A \) is measurable in \( \mathbb{R}^p \) and \( B \) is measurable in \( \mathbb{R}^q \). Note that \( A \times B = (A \times \mathbb{R}^q) \cap (\mathbb{R}^p \times B) \). Hence it suffices to prove that \( A \times \mathbb{R}^n \) is measurable (the proof for the set \( \mathbb{R}^p \times \mathbb{R}^q \) is similar).

The set \( A \) can be written as \( A = F \cup Z \) where \( F \) is a \( F_{\sigma} \) set and \( Z \) is of measure 0. Then 
\[ A \times \mathbb{R}^q = (F \times \mathbb{R}^q) \cup (N \times \mathbb{R}^q). \] Since \( F = \bigcup_{k} F_k \) with \( F_k \) closed, 
\[ F \times \mathbb{R}^q = \bigcup (F_k \times \mathbb{R}^q) \] so that \( F \times \mathbb{R}^q \) is \( F_{\sigma} \) set since \( F_k \times \mathbb{R}^q \) are closed. The set \( N \times \mathbb{R}^q \) has measure zero and the proof is given in (e).

5. Let \( N \) be a nonmeasurable subset of \( I = [0, 1] \) introduced in class. Prove:
   (a) If \( E \) is a measurable subset of \( N \), then \( m(E) = 0 \).
   (b) If \( E \subset \mathbb{R} \) satisfies \( m^*(E) > 0 \), then \( E \) contains a nonmeasurable subset.
   (c) If \( N^c = I \setminus N \), then \( m^*(N^c) = 1 \) and \( m^*(N^c \cup N) \neq m^*(N^c) + m^*(N) \).

Solution: (a) Assume that \( m(E) > 0 \), and consider sets \( E_r = E + r \) for \( r \in \mathbb{Q} = \mathbb{Q} \cap [-1,1] \). The sets \( E_r \) are measurable, disjoint since the sets \( E_r \subset N_r \) and the
sets $N_r$ are disjoint, and $\bigcup_{r \in Q} E_r \subset \bigcup_{r \in Q'} N_r \subset [-1, 2]$. Hence

$$m\left(\bigcup_{r \in Q'} E_r\right) = \sum_{r \in Q'} m(E_r) = \sum_{r \in Q'} m(E) \leq 3.$$ 

This implies that $m(E) = 0$, contradiction.

(b) Since $\sum_{r \in Q} N_r = \mathbb{R}$, one has $E = \bigcup_{r \in Q} E \cap N_r$. By countable sub-additivity of $m^*$, $m^*(E) \leq \sum_{r \in Q} m(E \cap N_r)$ so that there is $r \in Q$ for which $m^*(E \cap N_r) > 0$. If $m^*(E \cap N_r)$ is measurable, then since $E \cap N_r \subset N_r$, (a) implies that $m(E \cap N_r) = 0$, contradiction.

(c) Assume $m^*(N^c) < 1$. Take $0 < \varepsilon < 1 - m^*(N^c)$. Then there exists an open set $O$ such that $N^c \subset O$ and $m(O) < m^*(N^c) + \varepsilon < 1$. Then $O^c \subset N$. Since $O^c$ is measurable, (a) implies that $m(O^c) = 0$. However, $[0, 1] \subset O \cup O^c$ and $m(O) < 1$. So, $m(O^c) > 0$, contradiction. If $m^*(N \cup N^c) = m^*(N) + m^*(N^c)$, then $m^*(N) = 0$ which implies that $N$ is measurable, contradiction.