1. Let \((\varphi_n)\) be a Dirac sequence such that each \(\varphi_n\) is continuous and \(\text{supp } \varphi_n = \frac{B_{1/n}}{n}(0)\).

(a) Show that \((\varphi_n(x))\) converges to 0 for almost every \(x \in \mathbb{R}^d\). Deduce that \((\varphi_n)\) does not converge in \(L^1(\mathbb{R}^d)\).

(b) Deduce that \(L^1(\mathbb{R}^d)\) does not have a unity, that is, there is no \(g \in L^1\) such that \(f * g = f\) for all \(f \in L^1(\mathbb{R}^d)\).

2. Let \((\varphi_n)\) be a Dirac sequence such that each \(\varphi_n\) is continuous and \(\text{supp } \varphi_n = \frac{B_{1/n}}{n}(0)\). Show that if \(f\) is a continuous function on \(\mathbb{R}^d\), then the sequence \((f * \varphi_n)\) converges to \(f\) uniformly on every compact subset of \(\mathbb{R}^d\).

3. For \(f \in L^1(\mathbb{R}^d)\), define
\[
\hat{f}(y) = \int_{\mathbb{R}^d} e^{-i\langle x,y \rangle} f(x) dx \quad \text{for all } y \in \mathbb{R}^d.
\]
\(\hat{f}\) is called the Fourier transform of \(f\). Show that

(a) \(\hat{f} \in C_0(\mathbb{R}^d)\) and that \(\|\hat{f}\|_\infty \leq \|f\|_1\).

(b) If \(f, g \in L^1(\mathbb{R}^d)\), then \(f * g = \hat{f} \cdot \hat{g}\).

(c) Riemann-Lebesgue Lemma If \(f \in L^1(\mathbb{R}^d)\), then
\[
\lim_{|y| \to \infty} \int_{\mathbb{R}^d} e^{-i\langle x,y \rangle} f(x) dx = 0.
\]

\((\langle x, y \rangle)\) is the standard inner product in \(\mathbb{R}^d\), that is, \(\langle x, y \rangle = \sum_{i=1}^d x_i y_i\).

Hint: Show that if \(y \neq 0\), then
\[
F(f, y) = \int_{\mathbb{R}^d} e^{-i\langle x,y \rangle} f(x) dx = -F(\tau_{\pi y/|y|^2} f, y).
\]

Deduce that \(|2F(f, y)| \leq \|f - \tau_{\pi y/|y|^2} f\|_1\).