1. Let \( E \) be a Borel subset of \( \mathbb{R}^2 \). Prove that for every \( y \in \mathbb{R} \), the \( y \)-slice \( E^y \) is a Borel in \( \mathbb{R} \). *Hint:* Consider the collection \( C \) of subsets of \( \mathbb{R}^2 \) having the property that \( E^y \) is a Borel set in \( \mathbb{R} \) for every \( y \in \mathbb{R} \).

2. Let \( f(x, y) = xe^{-x^2(1+y^2)} \) for \( (x, y) \in I \times I \) where \( I = (0, \infty) \). Integrate \( f \) in two different ways and deduce that
\[
\int_{(0, \infty)} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]
This is one of the most important integrals.

3. Integrate the function \( f(x, y) = (\sin x) \cdot e^{-xy} \) over the set \( E = (0, a) \times (0, \infty) \).
   (a) Show that
   \[
   \int_{(0, a)} \frac{\sin x}{x} \, dx = \frac{\pi}{2} - (\cos a) \int_{(0, \infty)} \frac{e^{-ay}}{1 + y^2} \, dy - (\sin a) \int_{(0, \infty)} \frac{ye^{-ay}}{1 + y^2} \, dy.
   \]
   (b) Use (a) to show that
   \[
   \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \lim_{a \to \infty} \int_{0}^{a} \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
   \]
The left hand-side is the improper Riemann integral.
   (c) Show that \( f(x) = \frac{\sin x}{x} \) is not Lebesgue integrable on \( (0, \infty) \).

4. Let \( I = (0, 1) \) and let \( f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \) for \( (x, y) \in I \times I \). Calculate
   \[
   \int_{I} \left[ \int_{I} f(x, y) \, dx \right] \, dy \quad \text{and} \quad \int_{I} \left[ \int_{I} f(x, y) \, dy \right] \, dx.
   \]
   What is \( \int_{I \times I} |f| \)?

5. Let \( F \) be a closed subset of \( \mathbb{R} \) with \( m(F^c) < \infty \), and let \( \delta(x) = d(x, F) = \inf\{|x - y| : y \in F\} \).
   (a) Show that \( |\delta(x) - \delta(y)| \leq |x - y| \).
   Define
   \[
   M(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2} \, dy.
   \]
   (b) Show that \( M(x) = \infty \) if \( x \in F^c \) and \( M(x) < \infty \) for a.e. \( x \in F \).