8 Fourier Series

Our aim is to show that under reasonable assumptions a given 2π-periodic function f can be represented as convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) . \tag{8.1}$$

By definition, the convergence of the series means that the sequence $(s_n(x))$ of partial sums, defined by

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) ,$$

converges at a given point x to $f(x)$, $s_n(x) \to f(x)$.

We start with some definitions. A real valued function defined on $\mathbb{R}$ is said to be *periodic* with period L, or L-periodic, is $f(x+L) = f(x)$ for all x. In this case f is completely determined by its values on any interval $[a, a+L)$. A function f defined on $[a, a+L)$ can be extended in a unique way to a periodic function on $\mathbb{R}$. Indeed, given x, there exists a unique integer n so that x belongs to $[a+nL, a+(n+1)L)$ so that $x - nL \in [a, a+P)$ and we set $f(x) = f(x - nL)$. A function f on $[a, b]$ is *piecewise continuous* if f is continuous except a finite numbers of points and at each such point the one-sided limits $f(x^+) = \lim_{\varepsilon \to 0^-} f(x+\varepsilon)$ and $f(x^-) = \lim_{\varepsilon \to 0^-} f(x+\varepsilon)$ exist and are finite. We say that f is *piecewise smooth* (or *piecewise differentiable*) on $[a, b]$ if f and f' are piecewise continuous on $[a, b]$. If f is piecewise continuous, then it is Riemann integrable over any bounded interval contained in its domain. In addition, if f is L-periodic, then

$$\int_a^{a+L} f(x)\ dx = \int_0^L fx)\ dx$$

for every a. In what follows we focus on functions which are 2π-periodic.

Assume

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

We also assume that the convergence is well-behaved so that we can integrate term-by-term. We want to compute the coefficients a_k and b_k. We use integral identities

$$\int_{-\pi}^{\pi} \cos mx \cos nx \; dx = \begin{cases} 0, & m \neq n \\ 2\pi, & m = n = 0 \\ \pi, & m = n \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \; dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \; dx = 0.$$

To compute a_0, we multiply both sides of (8.1) by the constant function 1 and integrate over $[-\pi, \pi]$ and get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \; dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} 1 \; dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \; dx + b_n \int_{-\pi}^{\pi} \sin nx \; dx \right] \\ &= \frac{a_0}{2} \cdot (2\pi) = \pi a_0 \end{aligned}$$

To compute a_m with $m \geq 1$ we we multiply both sides of (8.1) by $\cos mx$ and integrate over $[-\pi, \pi]$ to get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx \; dx = &\frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx \; dx \\ &+ \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx \; dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \; dx \right] \\ &= a_m \int_{-\pi}^{\pi} \cos mx \cos mx \; dx = \pi a_m. \end{aligned}$$

Similarly, multiplying both sides of (8.1) by $\sin mx$ and integrating over $[-\pi, \pi]$, we find that

$$\int_{-\pi}^{\pi} f(x) \sin mx \; dx = \pi b_m.$$

Summing up,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx, \qquad n \geq 0 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx, \qquad n \geq 1. \end{aligned} \tag{8.2}$$

The coefficients a_n and b_n are called the *Fourier coefficients* of f and the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

the *Fourier series* of f. We denote this fact by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] .$$

The symbol $\sim$ should be read as f "has Fourier serier".

Example 8.1. Let f be a 2π-periodic function given by $f(x) = x$ for $x \in (-\pi, \pi]$. Then integrating by parts

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \; dx = \frac{1}{\pi} \left[\frac{x \sin nx}{n} \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} \sin nx \; dx = 0$$

and

$$\begin{aligned} b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \; dx &= \frac{1}{\pi} \left[-\frac{x \cos nx}{n} \right]_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos nx \; dx \\ &= -\frac{2 \cos n\pi}{n} = \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Consequently,

$$f(x) \sim 2 \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sin nx.$$

The series converges for every x in view of the following Dirichlet's test. *If (a_n) is a decreasing sequence converging to* 0 *and the sequence (s_n) of partial sums of the series $\sum b_n$ is bounded, then the series $\sum a_n b_n$ converges.* In our case, $a_n = \frac{1}{n}$ is decreasing and converges to 0, $(-1)^n \sin nx = \sin n(x + \pi)$, and

$$\sin \frac{\vartheta}{2} \cdot \sum_{k=1}^{n} \sin k\vartheta = \sin \frac{(n+1)\vartheta}{2} \cdot \sin \frac{n\vartheta}{2}$$

which implies that

$$\left| \sum_{k=1}^{n} \sin k\vartheta \right| \leq \frac{1}{|\sin \frac{\vartheta}{2}|}$$

if $\vartheta \neq 2\pi k$ for all $k \in \mathbb{Z}$.

Example 8.2. Consider f be a 2π-periodic function defined by $f(x) = |x|$ for $x \in [-\pi, \pi]$. Note that f is even. Since the product of even function with the odd function is odd, it follows that

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = 0.$$

Hence $b_n = 0$ for all $n \geq 1$. To compute b_n note that the product of an even function with an even function is even so that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\ dx.$$

If $n = 0$, then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\ dx = \pi,$$

and if $n \geq 1$, then integrating by parts, one finds that

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi} x\cos nx\ dx &= \frac{2}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin nx\ dx \\ &= \frac{2}{n^2\pi}\Big[\cos nx\Big]_0^{\pi} = \frac{2}{n^2\pi}\left[(-1)^n - 1\right].\end{aligned}$$

Hence $a_n = 0$ if n is even and $a_n = -\frac{4}{n^2\pi}$ when n is odd, and hence

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 1}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

Since the series $\sum \frac{1}{n^2}$ converges, the M-Weierstarss test implies that the above series converges.

In the above examples we have used the following identities,

$$\int_{-L}^{L} g(x)\ dx = \begin{cases} 2\int_0^L g(x)\ dx & \text{if } g \text{ is even} \\ 0 & \text{if } g \text{ is odd.}\end{cases}$$

Using that the sum of two even functions is even and the sum of two odd functions is odd, that the product of two even functions or two odd functions is even, and the product of an even function and an odd function is odd, we have the following proposition.

Proposition 8.3. *(i) If f is even, then its Fourier sine coefficients b_n are equal to* 0 *and f is represented by Fourier cosine series*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx \qquad \textit{for } n \geq 0.$$

(ii) If f is odd, then its Fourier cosine coefficients a_n are equal to 0 and f is represented by Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx \qquad \text{for } n \geq 1.$$

8.0.1 Convergence Theorem

Theorem 8.4. *Assume that f is a 2π-periodic function which is piecewise smooth. Then*

$$\frac{1}{2}\Big[f(x^-) + f(x^+)\Big] = \frac{a_0}{2} + \sum_{k=1}^{n}\left[a_k \cos kx + b_k \sin kx\right]$$

where $f(x^{\pm}) = \lim_{y\to x^{\pm}} f(y)$. In particular, if f is continuous at x, then

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left[a_k \cos kx + b_k \sin kx\right].$$

Example 8.5. Consider the function f from Example 8.1. Then f is smooth at all points except points $m\pi$ where m is odd. The one sided limits at these points are

$$f(m\pi^-) = \lim_{x\to m\pi^-} f(x) = \pi \qquad \text{and} \qquad f(m\pi^+) = \lim_{x\to m\pi^+} f(x) = -\pi$$

provided that m is odd. Consequently, the Fourier series converges of f converges to f at every point except points $m\pi$ with m odd. At these points the Fourier series converges to $\frac{1}{2}[f(m\pi^-) + f(m\pi^+)] = 0$. In particular,

$$\frac{x}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx, \qquad \text{for } x \in (-\pi, \pi).$$

Taking $x = \pi/2$, we find that

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}.$$

Example 8.6. Consider the function 2π-periodic function f given by $f(x) = |x|$ for $x \in (-\pi, \pi)$. (See Example 8.2.) Then f is continuous at every point and it is smooth except points $m\pi$ with m odd. Hence the Fourier series of f converges to f at every point. In particular,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \quad \text{for } x \in [-\pi, \pi].$$

Substituting $x = 0$, we find that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Set $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Then we note that

$$\frac{S}{4} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

and hence

$$\frac{3}{4}S = S - \frac{S}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

From this we conclude that

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Integration of Fourier series

We start with the following observation. Assume that f is 2π-periodic, piecewise continuous, and let $F(x) = \int_0^x f(y)\ dy$. Then F is 2π periodic if and only if $\int_{-\pi}^{\pi} f(y)dy = 0$. Indeed, we have

$$F(x+2\pi) - F(x) = \int_x^{x+2\pi} f(y)dy = \int_{-\pi}^{\pi} f(y)dy.$$

This means that the constant term in the Fourier series of f is equal to 0. Since $\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\ dy$, the number $a_0/2$ is the mean of the function f over the interval $[-\pi.\pi]$.

Theorem 8.7. *Assume that f is 2π-periodic and piecewise continuous and its mean is equal to 0. Then its Fourier series*

$$f(x) \sim \sum_{n\geq 1} [a_n \cos nx + b_n \sin nx]$$

can be integrated term by term and produce the Fourier series

$$F(x) = \int_0^x f(y)\ dy \sim C_0 + \sum_{n\geq 1} \left[-\frac{b_n}{n} \cos nx + \frac{a_n}{n} \sin nx \right]$$

where the constant $C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(y)\ dy$.

Example 8.8. Consider a 2π-periodic function f given by $f(x) = x$ on $(-\pi, \pi]$. (See Example 8.1.) Since f is odd, its mean value over $[-\pi, \pi]$ is equal to 0. Its Fourier series is given by

$$x = 2 \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sin nx, \qquad \text{for } x \in (-\pi, \pi].$$

Integrating term by term we find that

$$\frac{x^2}{2} = \frac{\pi^2}{6} - 2 \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^2} \cos nx \qquad \text{for } x \in (-\pi, \pi]$$

where $\pi^2/6 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2/2)\ dx$.

Differentiation of Fourier series

Proposition 8.9. *Assume that* f *is* 2π*-periodic, continuous, and piecewise smooth. Abbreviate by* a_n, b_n *the Fourier coefficients of* f *and by* a'_n, b'_n *the Fourier coefficients of* f'. *Then*

$$a'_n = nb_n \qquad \textit{and} \qquad b'_n = -na_n.$$

Proof. Integrating by parts,

$$a'_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx\ dx = \frac{1}{\pi} \Big[f(x) \cos nx \Big]_{-\pi}^{\pi} + \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\ dx = nb_n.$$

Similarly, $b'_n = -na_n$. ■

As a consequence, we get:

Theorem 8.10. *Let* f *be* 2π*-periodic, continuous, and piecewise smooth. In addition, assume that* f' *is piecewise smooth. If*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right],$$

then

$$\frac{1}{2} \Big[f'(x^-) + f'(x^+) \Big] = \sum_{n=1}^{\infty} \left[nb_n \cos nx - na_n \sin nx \right].$$

In particular, at points x *at which* $f'(x)$ *exists, the series converges to* $f'(x)$.

Proof. By Proposition 8.9,

$$f'(x) \sim \sum_{n=1}^{\infty} [nb_n \cos nx - na_n \sin nx],$$

and since f' is piecewise smooth, the theorem follows from Theorem 8.4. ∎

Example 8.11. Recall the function 2π-periodic function f given by $f(x) = |x|$ for $x \in (-\pi, \pi)$. (See Example 8.2 and Example 8.6.) The function f is continuous and piecewise smooth, and

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}, \qquad x \in [-\pi, \pi].$$

The derivative of f is given by

$$\frac{d}{dx}|x| = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

In view of Theorem 8.10,

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases}$$

At $x = 0$, the series converges to $\frac{1}{2}[f'(x^-) + f'(x^+)] = \frac{1}{2}[-1+1] = 0$.

8.0.2 Absolute and Uniform Convergence

Theorem 8.12. *Let f be 2π-periodic, continuous, and piecewise smooth. Then its Fourier series converges absolutely and uniformly.*

Proof. We prove the result under additional assumption that f' is piecewise smooth. This means that f'' is piecewise continuous. Abbreviate by a_n, b_n the Fourier coefficients of f and by a_n', b_n' the Fourier coefficients of f'. Since f is continuous,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

To prove the absolute convergence it suffices to show that $|a_n|, |b_n| \leq M/n^2$ for some constant M independent of n. Then, by the comparison test, $\sum_{n\geq 1} |a_n|$ and $\sum_{n\geq 1} |b_n|$ converge which imply the absolute convergence of the Fourier series. Also, the Weierstrass M-test implies the uniform convergence. By Proposition 8.9, $a_n = -b_n'/n$. That is,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = -\frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx.$$

By assumption, f' is piecewise continuous and so f'' is continuous except a finite number of point at which it has a jump. If (a, b) is an interval on which f'' is continuous, then

$$\begin{aligned}\frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\ dx &= \frac{1}{n^2\pi}\Big[f'(x)\cos nx\Big]_a^b - \frac{1}{n^2\pi}\int_a^b f''(x)\cos nx\ dx\\ &= \frac{1}{n^2\pi}\Big[f(b^-)\cos nx - f(a^+)\cos nx\Big] + \frac{1}{n^2\pi}\int_a^b f''(x)\cos nx\ dx.\end{aligned}$$

Since $|f'(x)|, |f''(x)| \leq K$ for some constant K and all x (at points x where f' and f'' has a jump discontinuity, $f'(x)$ one has to take left- and hand-side derivatives). So,

$$\frac{1}{n\pi}\left|\int_{-\pi}^{\pi} f'(x)\sin nx\ dx\right| \leq \frac{K(2+(b-a)}{n^2\pi} \leq \frac{2K(1+\pi}{n^2\pi} \leq \frac{4K}{n^2}.$$

Since f'' has a finite number of discontinuities, the integral $\int_{-\pi}^{\pi} f'(x)\sin nx\ dx$ can be written as the finite number, say m, of integrals $\int_a^b f'(x)\sin nx\ dx$ over intervals on which f'' is continuous. Then

$$|a_n| = \frac{1}{n\pi}\left|\int_{-\pi}^{\pi} f'(x)\sin nx\ dx\right| \leq \frac{4Km}{n^2}.$$

Similarly, $|b_n| \leq \frac{4Km}{n^2}$. ■

Theorem 8.13. *Let f be 2π-periodic. Assume that f is of class C^{k-1} and that $f^{(k-1)}$ is piecewise smooth. If the Fourier coefficients satisfy*

$$|a_n| \leq \frac{C}{n^{k+\alpha}} \qquad \text{and} \qquad |b_n| \leq \frac{C}{n^{k+\alpha}}$$

for some $\alpha > 1$ and $C > 0$, then f is of class C^k.

Proof. Let $a_n^{(j)}$ and $b_n^{(k)}$ are coefficients of the Fourier series of $f^{(j)}$. Then using Proposition 8.9 $\left|a_n^{(j)}\right| = n^j\,|a_n|$ if j is even and $\left|a_n^{(j)}\right| = n^k\,|b_n|$ if j is odd. Similarly, for $b_n^{(k)}$. Observe, using the assumption, that if $j \leq k$, then

$$\sum_{n\geq 1}\left|n^j a_n\right| \leq C\sum_{n\geq 1}\frac{1}{n^{k-j+\alpha}} \leq C\sum_{n\geq 1}\frac{1}{n^{\alpha}}$$

for $j \leq k$. By the Weierstrass M-test, the series $\sum_{n\geq 1}\left|n^j a_n\right|$ converges absolutely and uniformly for $j \leq k$. Similarly, the series $\sum_{n\geq 1}\left|n^j b_n\right|$ converges absolutely and uniformly for $j \leq k$ Hence, $f^{(j)} = \sum_{n\geq 1}[a_n^{j)}\cos nx + b_n^{j)}\sin nx]$ for $j \leq k$ and $f^{(k)}$ is continuous. ■

8.1 Changing a scale

So far we have considered 2π-periodic functions and their Fourier series over the interval $[-\pi, \pi]$. Now, given a function f defined on $[-L, L]$, we define $F(y) = f\left(\frac{L}{\pi}y\right)$. Then F is defined on $[-\pi, \pi]$ and its Fourier series is given by

$$F(y) \sim \frac{a_0}{2} + \sum_{n\geq 1}\big[a_n \cos ny + b_n \sin ny\big]$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ny \; dy \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\sin ny \; dy. \tag{8.3}$$

Since $f(x) = F(\frac{\pi}{L}x)$ for $x \in [-L, L]$, we deduce that

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}\big[a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x\big]$$

The coefficients a_n and b_n can be computed by integrating (8.3) by substitution $x = \frac{L}{\pi}y$. We get

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ny \; dy &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}y\right)\cos ny \; dy \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} \; dx \end{aligned} \tag{8.4}$$

and similarly

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\sin ny \; dy = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} \; dx. \tag{8.5}$$