MATH 404 ANALYSIS - SPRING 2008

SOLUTIONS to HOMEWORK 7

1. Let S be a bounded subset of $\mathbb{R}^n$ and let $f, g : S \to \mathbb{R}$ be integrable functions over S.

(a) Show that if $f(x) = g(x)$ for all $x \in S \setminus D$ where D has measure 0, then $\int_S f = \int_S g$.

(b) Show that if $f(x) \leq g(x)$ for all $x \in S$ and $\int_S f = \int_S g$, then f and g agree except on a set of measure zero.

Solution:
Let Q be a rectangle containing S. Denote by

$$f_S(x) = \begin{cases} f(x) & x \in S \\ 0 & x \notin S \end{cases} \quad \text{and} \quad g_S(x) = \begin{cases} g(x) & x \in S \\ 0 & x \notin S \end{cases}.$$

The fact that f and g are integrable means that F_S and G_S are integrable over Q.
(a) Since $f = g$ on $S \setminus D$, the functions f_S and g_S agree everywhere except the set D which is of measure 0. So $\int_Q f_S = \int_Q g_S$ and this means that $\int_S f = \int_S g$.
(b) Note that $f_S(x) \leq g_S(x)$ for all $x \in S$ and $f_S(x) = g_S(x)$ for $x \in Q \setminus S$. The equality $\int_S f = \int_S g$ means $\int_Q f_S = \int_Q g_S$. Hence $f_S(x) = g_S(x)$ for all $x \in Q \setminus D$ where D is a subset of Q of measure 0. Since $f_S = g_S$ outside of S, it follows that $D \subset S$ and that f and g agree except on the set D.

2. Let A be a rectangle in $\mathbb{R}^k$ and let B be a rectangle in $\mathbb{R}^n$. Set $Q = A \times B$ and assume that $f : Q \to \mathbb{R}$ is a bounded function. Show that if $\int_Q f$ exists, then

$$\int_B f(x, y)$$

exists for every $x \in A \setminus D$ where D is a set of measure zero in $\mathbb{R}^n$.

Solution:
By Fubini's theorem, the functions $\underline{I}(x) = \underline{\int}_B f(x, y)dy$ and $\overline{I}(x) = \overline{\int}_B f(x, y)dy$ are integrable over the rectangle A and $\int_Q f = \int_A \underline{I} = \int_A \overline{I}$. Since $\underline{I}(x) \leq \overline{I}(x)$ for all $x \in A$, it follows that $\underline{I}(x) = \overline{I}(x)$ for $x \in A \setminus D$ where D is a subset of A of measure 0. Hence for every $x \in A \setminus D$, the function $y \mapsto f(x, y)$ is integrable over B and so $\int_B f(x, y)$ exists for those values of x.

3. Let S be bounded subsets of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be a bounded function. Show that if f is integrable over S, then it is integrable over int S and $\int_S f = \int_{\text{int } S} f$.

Solution:
Abbreviate $A = \text{int } S$. Choose a rectangle Q such that $S \subset Q$ and let

$$f_S(x) = \begin{cases} f(x) & x \in S \\ 0 & x \notin S \end{cases} \qquad f_A(x) = \begin{cases} f(x) & x \in A \\ 0 & x \notin A \end{cases}.$$

Both functions f_S and $f_{\text{int } S}$ agree at points belonging to $Q \setminus \overline{S}$ and A. Clearly, they are continuous at every $x \in Q \setminus \overline{S}$. If $x \in A$, then f_S is continuous at x if and only if f_A is continuous at x. Since f is integrable over S, the function f_S is integrable over Q. Hence the set E of points where f_S fails to be continuous is of measure 0. The function f_S is equal to 0 outside of S hence it is continuous at every point belonging to the complement of $\overline{S}$. This means that the set D is a subset of $\overline{S}$. Write $D = E \cup F$ where $E \subset \partial S$ and $F \subset A$. Hence f_S and f_A are continuous at every point $x \in S \setminus B$. Assume that $x \in \partial A \setminus F$. Then f_S is continuous at x and $f_S(x) = 0$. By definition, $f_A(x) = 0$. Let (x_n) be any sequence of points in Q converging to x. Since $f_A(x_n)$ is equal either $f_S(x_n)$ or 0, and $f_S(x_n) \to f_S(x) = 0$, it follows that $f_A(x_n) \to 0 = f_A(x)$. Hence f_A is also continuous at x. This implies that f_A is continuous at every point except on the set $D = E \cup F$ which is of measure zero. This mean that f_A is integrable and that $\int_S f = \int_A f$.

4 Let S be a bounded subset of $\mathbb{R}^n$ that is the union the countable collection of rectifiable sets $S_1, S_2, \ldots$.

(a) Show that $S_1 \cup \cdots \cup S_n$ is rectifiable.
(b) Give an example showing that S need not be rectifiable.

Solution:
(a) Note that $\partial(S_1 \cup \cdots \cup S_n) \subset \partial S_1 \cup \cdots \cup \partial S_n$. Since each ∂S_i has measure zero, it follows that $\partial(S_1 \cup \cdots \cup S_n)$ has also measure zero.
(b) Let A be the set of rational numbers in $(0,1)$, say $A = \{r_i\}_{i \in \mathbb{N}}$. Choose $0 < a < 1$ and for each $i \in \mathbb{N}$, let (a_i, b_i) be an open subinterval of $(0,1)$ of length $b_i - a_i < a/2^i$ and containing r_i. $q_i \in (a_i, b_i)$. Set $S = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$. Then S is bounded and every (a_i, b_i) is rectifiable. However, ∂S does not have measure zero.

5. Let A and B be rectangles in $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively. Let S be a subset contained in $A \times B$. For each $y \in B$, define the *cross-section* of S by

$$S_y = \{x | \, x \in A \text{ and } (x,y) \in S\}.$$

Show that if S is rectifiable, and if S_y is rectifiable for every $y \in B$, then

$$v(S) = \int_B v(S_y).$$

Solution:
Since S is rectifiable, the constant function equal to 1 is integrable over S and $v(S) = \int_S 1 = \int_{A \times B} 1_S$. By Fubini's theorem, $\int_{A \times B} 1_S = \int_B \big[\underline{\int}_A 1_S(x,y)dx\big]dy$. Note that $I_S(x,y) = I_{S_y}(x)$. So, $\int_B \big[\underline{\int}_A 1_S(x,y)dx\big]dy = \int_B \big[\underline{\int}_A 1_{S_y}(x)dx\big]dy$. Since S_y is rectifiable for every $y \in B$, the function constant function 1 is integrable over S_y which means that $!_{S_y}$ is integrable over A for every $y \in B$. Hence $\underline{\int}_A I_{S_y}(x)dx = \int_A I_{S_y}(x)dx = \int_{S_y} 1dx = v(S_y)$ and $v(S) = \int_B \big[\underline{\int}_A 1_{S_y}(x)dx\big]dy = \int_B v(S_y)dy$ as claimed.