Problem 1. Let $f(x, y) = \frac{1}{(y+1)^2}$ and let

$A = \{(x, y) \in \mathbb{R}^2 | x > 0 \text{ and } x < y < 2x\}$,

$B = \{(x, y) \in \mathbb{R}^2 | x > 0 \text{ and } x^2 < y < 2x^2\}$.

Show: $\int_A f$ does not exist and $\int_B f$ exists. Find the value of $\int_B f$.

Solution: The set $A$. It suffices to find a sequence $(T_N)$ of compact Jordan measurable subsets $T_N$ of $A$ such that $A = \bigcup_{N \geq 1} T_N$ and $\int_{T_N} f \to \infty$. For example, let $T_N$ be a triangle bounded by the straight lines $y = x + 1/(2N)$, $y = 2x - 1/(2N)$ and the vertical line $y = N$. Then $T_N$ has vertices at the points $(1/N, 3/2N)$, $(N, N + 1/(2N))$ and $(N, 2N - 1/(2n))$. The triangle $T = T_N$ is contained in the rectangle $Q = I \times J$ where $I = [1/N, N]$ and $J = [3/(2N), 2N - 1/(2N)]$. Using the Fubini’s theorem,

$$\int_T f = \int_Q f_T = \int_1^N \left[ \int_{x+1/(2N)}^{2x-1/(2N)} \frac{1}{(y+1)^2} dy \right] dx$$

$$= \int_{1/N}^N \left[ \frac{1}{x + 1 + 1/(2N)} - \frac{1}{2x + 1 - 1/(2n)} \right] dx$$

$$= \ln(x + 1 + 1/(2N))|_{1/N}^N - \frac{1}{2} \ln(2x + 1 - 1/(2n))|_{1/N}^N$$

from which it follows that $\int_T f \to \infty$ as $N \to \infty$.

The set $B$. Take for example sets $P_N$ defined as follows. let $P_N$ the region bounded by the parabolas $y = x^2 + 1/(2N)$, $y = 2x^2 - 1/(2N)$ and the vertical line $y = \sqrt{N}$. This line intersects parabolas at $(\sqrt{N}, N + 1/(2N))$ and $\sqrt{N}, 2N - 1/(2N)$ and the parabolas intersect at the point $(1/\sqrt{N}, 3/(2N))$. The region $P_N$ is a subset of the rectangle $I \times J$ where $I = [1/\sqrt{N}, \sqrt{N}]$ and $J = [3/(2n), 2N - 1/(2N)]$. Clearly, $P_N$ is compact Jordan measurable and $B = \bigcup_{N \geq 1} P_N$. Write $P = P_N$. Then by
the Fubini’s theorem,
\[ \int f \, dx = \int_{\mathbb{R}} \int f \, dy = \int_{1/\sqrt{N}}^{\sqrt{N}} \left[ \frac{1}{x^2 + 1 + 1/(2N)} - \frac{1}{2x^2 + 1 - 1/(2N)} \right] \left[ \frac{1}{\sqrt{1 + 1/(2N)}} \arctan \left( \frac{x}{\sqrt{1 + 1/(2N)}} \right) \right]^{\sqrt{N}}_{1/\sqrt{N}} \]
\[ = \frac{1}{\sqrt{1 + 1/(2N)}} \arctan \left( \frac{\sqrt{2x}}{\sqrt{1 - 1/(2N)}} \right) \left|^{\sqrt{N}}_{1/\sqrt{N}} \right. \]

Evaluating the function at the end points and taking the limit as \( N \to \infty \) we get that \( \int f \) converges to \((1 - \sqrt{2})\pi/2\). So \( f \) is integrable over \( B \) and \( \int_B f = (1 - \sqrt{2})\pi/2 \).

**Problem 2.** Let \( U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \) and let \( f(x, y) = \frac{1}{(x^2 + y^2)^{1/2}} \) for \((x, y) \neq (0, 0)\). Determine if the function \( f \) is integrable over \( U \setminus \{(0,0)\} \) and over \( \mathbb{R}^2 \setminus U \).

**Solution:** Let \( A_{a,b} = \{(x, y) \in \mathbb{R}^2 \mid a < |(x, y)| < b\} \). For \( A = (a, b) \times (0, 2\pi) \), let \( g : A \to B = g(A) \) be defined by \( g(r, \varphi) = (r \cos \varphi, r \sin \varphi) \). Then \( B = A_{a,b} \setminus L \) where \( L = \{(x, y) \mid y = 0 \text{ and } x \geq 0\} \). Moreover, \( g \) is one-to-one on \( A \) and \( Dg(r, \varphi) = r > 0 \) so that \( g \) is a diffeomorphism between \( A \) and \( B \). By the change of variables theorem (using the fact that \( f \) is continuous for \((x, y) \neq 0\), that the boundary of \( A_{a,b} \) and \( L \) have measure 0 in \( \mathbb{R}^2 \)) and the Fubini’s theorem,

\[ \int_{A_{a,b}} f = \int_{A_{a,b}} f = \int_{B} f = \int_{g(A)} f = \int_{A} (f \circ g) \, |\det Dg| \]
\[ = \int_{A} \frac{1}{r^2} d\varphi = 2\pi \int_{a}^{b} \frac{1}{r^2} dr = 2\pi \left( \frac{1}{a} - \frac{1}{b} \right) \]

**The set \( U \)**. Let \( C_N = \{(x, y) \in \mathbb{R}^2 \mid 1/N \leq |(x, y)| \leq 1 - 1/N \} \), then \( C_N \) is compact, Jordan measurable, and \( U = \bigcup_{N \geq 1} C_N \). Then by the above calculations

\[ \int_{C_N} f = 2\pi \left( \frac{1}{1/N} - \frac{1}{1 - 1/N} \right) = 2\pi \left( N - \frac{N}{N - 1} \right) \to \infty \]

showing that \( f \) is not integrable on \( U \).

**The set \( \mathbb{R}^2 \setminus U \)**. Consider \( C_N = \{(x, y) \mid 1 + 1/N \leq |(x, y)| < N\} \). Then \( C_N \) is compact, Jordan measurable, and \( \mathbb{R}^2 \setminus U = \bigcup_{N \geq 1} C_N \), and

\[ \int_{C_N} f = 2\pi \left( \frac{1}{1 + 1/N} - \frac{1}{N} \right) = 2\pi \left( \frac{N}{1 + N} - \frac{1}{N} \right) \to 1 \]

so that \( f \) is integrable on \( \mathbb{R}^2 \setminus U \) and \( \int_{\mathbb{R}^2 \setminus U} f = 1 \).

**Problem 3.** Let \( \pi_k : \mathbb{R}^n \to \mathbb{R} \) be the projection onto the \( k \)th factor, i.e., \( \pi_k(x) = x_k \).
If \( S \) is a bounded Jordan-measurable subset of \( \mathbb{R}^n \) with non-zero volume, define the centroid \( c(S) \) of \( S \) to be the point in \( \mathbb{R}^n \) whose \( k \)th coordinate is equal to

\[ c(S)_k = \frac{1}{v(S)} \int_s \pi_k \]
(a) $S$ is said to be symmetric with respect to the subspace $x_k = 0$ if $g(S) = S$ where $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$g(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, -x_k, x_{k+1}, \ldots, x_n).$$

Show that if $S$ is symmetric with respect to the subspace $x_k = 0$, then $c(S)_k = 0$.

(b) Let $U = \{(x, y, z) \in \mathbb{R}^3 | z > 0$ and $x^2 + y^2 + z^2 < r^2\}$. Use the spherical coordinates and the change of variables theorem to compute $c(U)$.

**Solution:** (a) This follows from the change of variables theorem. First note that $\pi_k \circ g = \pi_k$ and that $Dg(x)$ is diagonal matrix with all entries equal to 1 except the $k$th diagonal entry is equal to $-1$. So, $|Dg| = 1$ and

$$c(S)_k = \frac{1}{v(S)} \int_S \pi_k = \frac{1}{v(S)} \int_{g(S)} \pi_k = \frac{1}{v(S)} \int_S (\pi_k \circ g) |\det Dg| = -\frac{1}{v(S)} \int_S \pi_k = -c(S)_k.$$

Hence $c(S)_k = 0$ as claimed.

(b) The set $U$ is symmetric with respect to subspaces $x = 0$ and $y = 0$. Hence $c(U)_x = 0$ and $c(U)_y = 0$. To calculate $c(U)_z$ we use the change of variables theorem and the spherical coordinates. Let $A = (0, r) \times (0, \pi/2) \times (0, 2\pi)$ and

$$g : A \to \mathbb{R}^3 \text{ be defined by } g(r, \varphi, \psi) = (\rho \sin \varphi \cos \psi, \rho \sin \varphi \cos \psi, \rho \cos \varphi).$$

Then $g(A) = B := \{(x, y, z) | x^2 + y^2 + z^2 < r^2$ and $z > 0\} \setminus L$, where $L = \{(x, y, z) \in \mathbb{R}^3 | y = 0, x \geq 0\}$, is open, $g : A \to B$ is one-to-one and $Dg(\rho, \varphi, \psi) = \rho^2 \sin \varphi > 0$ on $A$. So, $g$ is a diffeomorphism. Using $v(U) = v(B^3(r))/2 = 2\pi r^3/3$ (see Problem 5), the change of variables theorem, we find that

$$c(U)_z = \frac{1}{v(U)} \int_U \pi_z = \frac{3}{2\pi r^3} \int_A (\pi_z \circ g) |\det Dg| = \frac{3}{2\pi r^3} \int_A \rho^3 \cos \varphi \sin \varphi$$

$$= \frac{3 \cdot 2\pi \cdot r^4}{4 \cdot 2\pi} \int_0^{\pi/2} \cos \varphi \sin \varphi d\varphi = \frac{3r}{8}.$$
Problem 4. Let $A$ be an bounded open Jordan-measurable subset of $\mathbb{R}^{n-1}$ and let $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ with $p_n > 0$. Define the subset $S$ of $\mathbb{R}^n$ by setting

$$S = \{x \in \mathbb{R}^n \mid x = (1-t)a + tp \text{ where } t \in (0,1) \text{ and } a \in A \times \{0\}\}.$$  

(a) Define a diffeomorphism $g$ between $S$ and $A \times (0,1)$.

(b) Calculate the volume $v(S)$ in terms of the volume $v(A)$ of $A$.

(c) Express the centroid $c(S)$ in terms of the centroid of $c(A)$ and $p$ and show that $c(S)$ lies on the line segment connecting $(c(A),0)$ and $p$.

Solution: (a) We write $a = (b,0)$ with $b \in A$ for a point $a \in A \times \{0\}$. Then define $g : A \times (0,1) \to S$ by $g(b,t) = (1-t)(b,0) + tp = (1-t)a + tp$. Clearly, $g$ is onto. If $g(b,t) = g(b,s)$, then $(1-t)(b,0) + tp = (1-s)(b,0) + sp$. Then $tp_n = sp_n$ implying that $t = s$ and this in turn implies that $b = \tilde{b}$. So, $g$ is one-to-one. $g$ is also of class $C^\infty$. Moreover,

$$Dg(b,t) = \begin{bmatrix}
1 - t & 1 - t & \ldots & 1 - t & p_1 - a_1 \\
1 - t & 1 - t & \ldots & 1 - t & p_2 - a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 - t & 1 - t & \ldots & 1 - t & p_{n-1} - a_{n-1} \\
0 & 0 & \ldots & 0 & p_n
\end{bmatrix}$$

so that $\det Dg(b,t) = (1-t)^{n-1}p_n > 0$. Consequently, the inverse $g^{-1}$ is also of class $C^\infty$. Summing up, $g$ is a $C^\infty$-diffeomorphism.

(b) $v(S) = \int_S 1 = \int_{g(A\times(0,1))} 1 = \int_{A\times(0,1)} |\det Dg|$

$$= \int_{A\times(0,1)} (1-t)^{n-1}p_n = p_nv(A) \int_0^1 (1-t)^{n-1} = \frac{p_n}{n}v(A).$$

(c) Using the change of the variables theorem and Fubini’s theorem, we calculate for $1 \leq k \leq n - 1$,

$$c(S)_k = \frac{1}{v(S)} \int_S \pi_k = \frac{1}{v(S)} \int_{g(A\times(0,1))} \pi_k = \frac{1}{v(S)} \int_{A\times(0,1)} \pi_k \circ g |\det Dg|$$

$$= \frac{1}{v(S)} \int_{A\times(0,1)} [(1-t)a_k + tp_k](1-t)^{n-1}p_n$$

$$= \frac{n}{v(A)} \int_{A\times(0,1)} (1-t)^n a_k + \frac{n}{v(A)} \int_{A\times(0,1)} t(1-t)^{n-1}p_k$$

$$= \frac{n}{n+1}c(A)_k + n \left(\frac{1}{n} - \frac{1}{n+1}\right)p_k = \frac{n}{n+1}c(A)_k + \left(1 - \frac{n}{n+1}\right)p_k$$

and

$$c(S)_n = \frac{1}{v(S)} \int_S \pi_n = \frac{1}{v(S)} \int_{g(A\times(0,1))} \pi_n = \frac{1}{v(S)} \int_{A\times(0,1)} \pi_n \circ g |\det Dg|$$

$$= \frac{n}{v(A)} \int_{A\times(0,1)} t(1-t)^{n-1}p_n = \left(1 - \frac{n}{n+1}\right)p_n$$

Hence

$$c(S) = \frac{n}{n+1}(c(A),0) + \left(1 - \frac{n}{n+1}\right)p.$$
Problem 5. Let $B^n(r)$ be a closed ball in $\mathbb{R}^n$ of radius $r$ and centered at 0.

(a) Show that $v(B^n(r)) = \alpha_n r^n$ where $\alpha_n = v(B^n(1))$.
(b) Calculate $\alpha_1$ and $\alpha_2$.
(c) Compute $\alpha_n$ in terms of $\alpha_{n-2}$.
(d) Find the formula for $\alpha_n$ by considering two cases: $n$ is odd and $n$ is even.

Solution: (a) Consider $g : B^1(1) \rightarrow \mathbb{R}^n$ defined by $g(x) = rx$. Then $g(B^1(1)) = B^n(r)$ so that $g$ is a diffeomorphism between $B^1(1)$ and $B^n(r)$ with $Dg(x) = r$. By the change of variables theorem,

$$v(B^n(r)) = \int_{B^n(r)} 1 = \int_{g(B^1(1))} 1 = \int_{B^1(1)} |\det Dg| = \int_{B^1(1)} r^n = r^n B^n(1).$$

(b) Since $B^1(1) = (-1, 1)^n$, $v(B^1(1)) = 2$. Let $A = (0, 1) \times (0, 2\pi)$ and $g : A \rightarrow \mathbb{R}^2$ be defined by $g(r, \phi) = (r \cos \phi, r \sin \phi)$. Then $g$ is one-to-one, $B := g(A) = B^2(1) \setminus I$ where $I = \{(x, y) | x \geq 0 \text{ and } y = 0\}$. So $g(A)$ is open. In addition, $Dg(r, \phi) = r > 0$ on $A$. So $g$ is a diffeomorphism between $A$ and $B$ and by the change of variables theorem and Fubini’s theorem,

$$\int_{B^2(1)} 1 = \int_A |\det Dg| = \int_A r = \int_{(0,2\pi)} \left[ \int_{(0,1)} r ~d\phi \right] dr = \pi.$$

(c) For $z \in \mathbb{R}^n$ write $z = (x, y)$ where $x \in \mathbb{R}^{n-2}$ and $y \in \mathbb{R}^2$. Now if $z \in B^n(1)$, then $|z|^2 = |x|^2 + |y|^2 < 1$. Hence $|y| < 1$ implying that $y \in B^2(1)$ and $|x|^2 < 1 - |y|^2$ which implies that $x \in B^{n-2}(\sqrt{1 - |y|^2})$. Next note that $B^n(1) \subset Q \times I$ where $Q = [-1,1]^{n-2}$ and $I = [-1,1]^2$.

For every $y \in I^2$, then $\chi_{B^n(1)}(x, y) = 1$ for $y \in B^2(1)$ and if $y \notin B^2(1)$, then $\chi_{B^n(1)}(x, y) = 0$. So, $\chi_{B^n(1)}(\cdot, y)$ is integrable over $Q$ and

$$\int_Q \chi_{B^n(1)}(x, y) dx = \begin{cases} 0 & y \notin B^2(1), \\ \left(1 - |y|^2\right)^{\frac{n-2}{2}} \alpha_{n-2} & y \in B^2(1). \end{cases}$$

So by the Fubini’s theorem

$$\int_Q \chi_{B^n(1)} = \int_{I^2} \left[ \int_Q \chi_{B^n(1)}(x, y) dx \right] dy = (1 - |y|^2)^{\frac{n-2}{2}} \alpha_{n-2} \chi_{B^2(1)}$$

The integral on the right-hand side can be calculated using the change of variables theorem using $g : A = (0, 1) \times (0, 2\pi) \rightarrow \mathbb{R}^2$, $g(r, \phi) = (r \cos \phi, r \sin \phi)$. We have

$$\int_{B^2(1)} (1 - |y|^2)^{\frac{n-2}{2}} = \int_A (1 - r^2)^{n-2} dr d\phi = \pi \int_0^1 (1 - r^2)^{\frac{n-2}{2}} (2r) dr = \frac{2\pi}{n}.$$ 

So,

$$\alpha_n = \frac{2\pi}{n} \alpha_{n-2}.$$
(d) From (c) and \( \alpha_1 = 2 \) and \( \alpha_2 = \pi \) it follows that

\[
\alpha_{2k} = \frac{\pi^k}{k!} \quad \text{and} \quad \alpha_{2k+1} = \frac{2^{k+1}}{1 \cdot 3 \cdots (2k + 1)} \pi^k
\]