

MATH 404 ANALYSIS - SPRING 2008

SOLUTIONS to HOMEWORK 6

1. Let Q be a closed rectangle in \mathbb{R}^n and let $f : Q \rightarrow \mathbb{R}$ be continuous. Show that f is Riemann integrable.

Solution:

To simplify things we use the norm $\|x\| = \max\{|x_1|, \dots, |x_n|\}$ on \mathbb{R}^n and write $Q = I_1 \times \dots \times I_n$ where I_i is a closed interval. It suffices to show that given $\varepsilon > 0$ there is a partition P of Q such that $U(f, P) - L(f, P) < \varepsilon/(2v(Q))$. Since Q is compact and f is continuous, f is uniformly continuous. Hence for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $x, y \in Q$ and $\|x - y\| < \delta$. Divide each interval I_i into a finite number of subintervals of length $< \delta$. Denoting by P_i a partition of I_i consisting of end points of these subintervals of I_i , set $P = (P_1, \dots, P_n)$. If R is any subrectangle determined by P , then $\|x - y\| < \delta$ for every $x, y \in R$. Hence $|f(x) - f(y)| < \varepsilon/(2v(Q))$ for every $x, y \in R$. From this one concludes that $0 \leq M_R(f) - m_R(f) \leq \varepsilon/(2v(Q))$ so that

$$U(f, P) - L(f, P) = \sum_R [M_R(f) - m_R(f)] \cdot v(R) \leq \frac{\varepsilon}{2v(Q)} \sum_R v(R) = \frac{\varepsilon}{2v(Q)} \cdot v(Q) = \frac{\varepsilon}{2}.$$

2. Let $Q = [0, 1] \times [0, 1]$ and let $f : Q \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 0 & x \neq y, \\ 1 & x = y. \end{cases}$$

Show that f is Riemann integrable over Q .

Solution:

It suffices to show that given $\varepsilon > 0$ there is a partition P of Q such that $U(f, P) - L(f, P) < \varepsilon$. Let $n \in \mathbb{N}$ and let $P_1 = \{0 = t_0 < t_1 < \dots < t_n = 1 \text{ where } t_i = i/n \text{ for } i = 0, \dots, n\}$. Set $P_2 = P_1$ and $P = (P_1, P_2)$. Let R be a subrectangle of Q determined by the partition P . Then $M_R(f) = m_R(f) = 0$ if R does not have a vertex of the form $(i/n, i/n)$, otherwise $M_R(f) = 1$ and $m_R(f) = 0$. Hence

$$U(f, P) - L(f, P) = \sum_R v(R)$$

where the sum is taken over rectangle R having at least one vertex of the form $(i/n, i/n)$ for some $0 \leq i \leq n$. Given $0 \leq i \leq n$, there are at most four subrectangles R having a vertex at $(i/n, i/n)$. So, $U(f, P) - L(f, P) = \sum_R v(R) \leq 4(n+1) \frac{1}{n^2} < \varepsilon$ if n is large.

3. Let $f : [a, b] \rightarrow \mathbb{R}$. The graph of f is the subset

$$G(f) = \{(x, y) \in [a, b] \times \mathbb{R} \mid y = f(x)\}$$

of \mathbb{R}^2 . Show that if f is continuous, then G_f has measure zero in \mathbb{R}^2 .

Solution:

Since f is defined on a closed and bounded interval and it is continuous, it is uniformly continuous. Hence given $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that $|f(x) - f(y)| < \varepsilon/2(b-a)$ for $x, y \in [a, b]$ satisfying $|x - y| < 1/n$. Divide the interval $[a, b]$ into

a finite number of subintervals I_i , $i = 1, \dots, N$, each of length $< 1/n$. Choose some point $x_i \in I_i$. Then $|f(x) - f(x_i)| < \varepsilon/2(b-a)$ for every $x \in I_i$. Hence $f(x) \in (f(x_i) - \varepsilon/2(b-a), f(x_i) + \varepsilon/2(b-a))$. Then the graph of f over the interval I_i is contained in the rectangle $R_i = I_i \times [f(x_i) - \varepsilon/2(b-a), f(x_i) + \varepsilon/2(b-a)]$ whose area is $< |I_i| \cdot \varepsilon/(b-a)$. Since $G(f) = \bigcup_{i=1}^n \{(x, y) | x \in I_i, y = f(x)\}$, it follows that $G(f)$ can be covered by rectangles R_i whose total area is equal to $\sum_{i=1}^N v(R_i) < \frac{\varepsilon}{b-a} \sum_{i=1}^N |I_i| = \varepsilon$.

4. Let Q be a closed rectangle in \mathbb{R}^n and let $f : [Q \rightarrow \mathbb{R}$ be a bounded function. Show that if f vanishes except on a closed set B having measure zero, then $\int_Q f$ exists and is equal to 0.

Solution:

Write $Q = I_1 \times \dots \times I_n$. Since f is bounded, $|f(x)| \leq M$ for $x \in Q$. Let $\varepsilon > 0$. Since B has measure zero, there is a sequence (Q_k) of closed rectangle such that

$$B \subset \bigcup_{k=1}^{\infty} \text{int } Q_k \quad \text{and} \quad \sum_{k=1}^{\infty} v(Q_k) < \frac{\varepsilon}{M}$$

Since B is closed and bounded (it is contained in Q), it is compact. So, there is $N \in \mathbb{N}$ such that $B \subset \bigcup_{k=1}^N Q_k$. Denote by P_i the partition of the interval I_i by taking the end points of I_i and end points of the i th components of the rectangles Q_k for $k = 1, \dots, N$. The let $P = (P_1, \dots, P_n)$. If R is the subrectangle determined by P , we have that $M_R(f) = 0$ if $R \neq Q_k$ for $k = 1, \dots, N$ so that

$$U(f, P) = \sum_R M_R(f)v(R) = \sum_{k=1}^N M_{Q_k}(f)v(Q_k) \leq M \sum_{k=1}^N v(Q_k) \leq M \frac{\varepsilon}{M} = \varepsilon.$$

Hence $U(f, P) - L(f, P) \leq U(f, P) < \varepsilon$ showing that f is integrable. This also implies that $\int_Q f \leq U(f, P) < \varepsilon$. Since ε is arbitrary positive number, $\int_Q f = 0$ as claimed.

5. Let Q be a closed rectangle in \mathbb{R}^n and let $f : [Q \rightarrow \mathbb{R}$ be a bounded function. Assume that f is integrable over Q .

- (a) Show that if $f(x) \geq 0$ for all $x \in Q$, then $\int_Q f \geq 0$.
- (b) Show that if $f(x) > 0$ for all $x \in Q$, then $\int_Q f > 0$.

Solution:

(a) Let P be any partition of Q . Then $L(f, P) \geq 0$ and so the lower integral of f over Q is nonnegative. Since $\int_Q f$ exists, it is equal to the lower integral of f so that $\int_Q f \geq 0$.

(b) Since f is integrable, the set of point D where f is discontinuous is of measure zero. Hence there is x_0 in Q at which f is continuous. Then, by assumption, $f(x_0) > 0$ and there is a subrectangle $Q' \subset Q$ containing x_0 such that $f(x) > f(x_0)/2$ for all $x \in Q'$. Let P be any partition of Q and let \mathcal{R} be the collection of subrectangles R determined by P which intersect Q' . Then $Q' \subset \bigcup_{R \in \mathcal{R}} R$ and so $\sum_{R \in \mathcal{R}} v(R) \geq v(Q')$. Hence

$$L(f, P) = \sum_R m_R(f)v(R) \geq \sum_{R \in \mathcal{R}} m_R(f)v(R) \geq \frac{f(x_0)}{2} \sum_{R \in \mathcal{R}} v(R) \geq \frac{f(x_0)}{2} v(Q').$$

Hence $\int_Q f \geq \frac{f(x_0)}{2}v(Q)$.