MATH 404 ANALYSIS - SPRING 2008

SOLUTIONS to HOMEWORK 5

1. Abbreviate by $\mathbb{R}_+ = (0, \infty)$. Define $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by

$$f(x, y) = (x + y, \frac{x}{y})$$

(a) Show that f is one-to-one and $f(\mathbb{R}^2_+) = \mathbb{R}^2_+$.

(b) Find the inverse $g : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ of f and show that f and g are of class C^∞.

Solution:
(a) If $f(x_1, y_1) = f(x_2, y_2)$, then (1), $x_1 + y_1 = x_2 + y_2$ and (2) $x_1/y_1 = x_2/y_2$. From (2), $x_1 = x_2(y_1/y_2)$. Hence (1) becomes, $x_2(y_1/y_2) + y_1 = x_2 + y_2$. This gives $x_2)(y_1/y_2) - 1) = y_2 - y_1 1$, that is, $(x_2/y_2)(y_1 - y_2) = y_2 - y_1$. If $y_1 \neq y_2$, then we can divide both sides by $y_1 - y_2$ to get $x_2/y_2 = -1$, i.e., $x_2 = -y_2$ which is impossible since $x_2, y_2 > 0$. So $y_1 = y_2$ and then (1) implies that $x_1 = x_2$. So, f is one-to-one.Let $(u, v) \in \mathbb{R}^2_+$. We will show that there is $(x, y) \in \mathbb{R}^2_+$ such that (1) $x+y = u$ and (2) $x/y = v$. From (2), $x = vy$ and (1) becomes, $vy+y = y(v+1) = u$. So $y = u/(1 + v)$ and $x = uv/(1 + v)$. Hence $f(\mathbb{R}^2_+) = \mathbb{R}^2_+$. The inverse is given by $g(u, v) = (u/(1 + v), uv/(1 + v))$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $f(x) = \|x\|^2 \cdot x$. Show that f is of class C^∞ and that f carries the unit ball $B_1(0)$ onto itself in a one-to-one fashion. Show that the inverse function of f is not differentiable at 0.

Solution:
Assume that $f(x) = f(y)$. Then $\|x\|^2 x = \|y\|^2 y$. Hence $\left\|\|x\|^2 x\right\| = \|x\|^3 = \left\|\|y\|^2 y\right\| = \|y\|^3$ showing that $\|x\| = \|y\|$. Since $\|x\|^2 x = \|y\|^2 y$, one finds that $x = y$. So, f is one-to-one. Let $\|u\| < 1$. Then there is $x \in B_1(0)$ such that $f(x) = u$. Indeed, if $u = 0$, then $x = 0$. Otherwise, one has to have $\left\|\|x\|^2 x\right\| = \|x\|^3 = \|u\|$ so that $\|x\| = \|u\|^{1/3}$ and $x = u/\|u\|^{2/3}$. Hence $f(B_1(0)) = B_1(0)$ and the inverse function $g : B_1(0) \to B_1(0)$ of f is given by $g(0) = 0$ and $g(u) = u/\|u\|^{2/3}$. The ith component g_i of g is equal to $g_i(0) = 0$ and $g_i(u) = u_i/\|u\|^{2/3} = u_1/[\sum_{j=1}^n u_j^2]^3$. Then

$$\frac{g_i(0 + te_i) - g_i(0)}{t} = \frac{t}{t \cdot t^{2/3}} = \frac{1}{t^{2/3}} \to \infty \quad \text{as } t \to 0.$$

Hence the partial derivative $D_i g_i(0)$ does not exist and so, g is not differentiable at 0.

3. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$g(x, y) = (2ye^{2x}, xe^y),$$

and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x, y) = (3x - y^2, 2x + y, xy + y^3).$$

(a) Show that there is a neighborhood of $(0, 1)$ that g carries in a one-to-one fashion onto a neighborhood of $(2, 0)$.

(b) Find $D(f \circ g^{-1})$ at $(2, 0)$.

Solution:
(a) The function g is differentiable with

$$Dg(x,y) = \begin{bmatrix} 4ye^{2x} & 2e^{2x} \\ e^{y} & xe^{y} \end{bmatrix} \quad \text{so that} \quad Dg(0,1) = \begin{bmatrix} 4 & 2 \\ e & 0 \end{bmatrix}.$$

So, det$DF(0,1) = -2e \neq 0$. By the inverse function theorem there is an open neighborhood U of the point $(1,0)$ such that $g(U)$ is an open neighborhood of the point $g(0,1) = (2,0)$ such that g is one-to one on U.
(b) Let $g^{-1} : g(U) \to U$ be the inverse of g. Then by the chain rule,

$$D(f \circ g^{-1})(2,0) = Df(g^{-1}(2,0))Dg^{-1}(2,0) = Df(0,1)[Dg(0,1)]^{-1}.$$

Since

$$Df(x,y) = \begin{bmatrix} 3 & -2y \\ 2 & 1 \\ y & x+3y^2 \end{bmatrix} \quad \text{so that} \quad Df(0,1) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix},$$

one gets

$$D(f \circ g^{-1})(2,0) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ e & 0 \end{bmatrix}^{-1} = \frac{1}{2e} \begin{bmatrix} 3 & -2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ e & -4 \end{bmatrix} = \frac{1}{2e} \begin{bmatrix} -2e & 18 \\ e & 0 \\ 3e & -10 \end{bmatrix}$$

4. Let A be open in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^n$ be of class C^r. Assume that $Df(x)$ is non-singular for every $x \in A$. Show that even if f is not one-to-one on A, the set $B = f(A)$ is open in $\mathbb{R}^n$.

Solution:
Let $b \in B = f(A)$. We have to show that there is $\delta > 0$ such that $B_\delta(b) \subset B$. Let $a \in A$ be such that $b = f(a)$. Since $Df(a)$ is non-singular, we find $\varepsilon > 0$ such that $\overline{B}_\varepsilon \subset A$ and f is one-to-one on $\overline{B}_\varepsilon(a)$. The boundary $\partial B_\varepsilon(a)$ is compact so that also $f(\partial B_\varepsilon(a))$ is compact and $b \notin f(\partial B_\varepsilon(a))$. Set $\delta = (1/2)d(f(\partial B_\varepsilon(a)), b)$. Then $\delta > 0$. We claim that if $c \in B_\delta(b)$, then there is $y \in B_\varepsilon(a)$ such that $c = f(y)$ (which shows that $B_\delta(b) \subset B = f(A)$. To see this consider $\varphi : \overline{B}_\varepsilon(a) \to \mathbb{R}$ defined by $\varepsilon(x) = \|f(x) - c\|^2$. Since φ is continuous and defined on a compact set there is $y \in ovB_\varepsilon(a)$ such that $\varphi(y) = \min\{\varphi(x) | x \in ovB_\varepsilon(a)\}$. At $x = a$, $\varphi(a) = \|f(a) - c\|^2 = \|b - c\|^2 < \delta^2$. Hence $\varphi(y) < \delta^2$, If $\|y\| = \varepsilon$, then $\|f(y) - c\| = \|f(y) - b + b - c\| \geq \|f(y) - b\| - \|b - c\| \geq 2\delta - \delta \geq \delta$. So in this case $\varphi(y) \geq \delta^2$ contradicting $\varphi(y) < \delta^2$. So $y \in B_\varepsilon(a)$ and $D\varphi(y) = 0$. Since $\varphi(x) = \sum_{j=1}^n (f_j(x) - c_j)^2$, one gets that

$$D_i\varphi(y) = 2\sum_{j=1}^n (f_j(y) - c_j)D_i f_j(y)$$

so that

$$\begin{aligned} D\varphi(y) &= \begin{bmatrix} 0 & 0 & \cdots 0 & 0 \end{bmatrix} \\ &= 2\begin{bmatrix} \sum_{j=1}^n (f_j(y) - c_j)D_1 f_i(y) & \cdots & \sum_{j=1}^n (f_j(y) - c_j)D_n f_j(y) \end{bmatrix} \\ &= 2\begin{bmatrix} f_1(y) - c_1 & \cdots & f_n(y) - c_n \end{bmatrix} \cdot Df(y) \end{aligned}$$

Since $Df(y)$ is non-singular, it follows that

$$\begin{bmatrix} f_1(y) - c_1 & \cdots & f_n(y) - c_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix},$$

that is, $f(y) = c$. Hence the claim follows.

5. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be of class C^2 with $F(0,0) = 0$ and $DF(0,0) = \begin{bmatrix} 2 & 3 \end{bmatrix}$. Let $G : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$G(x,y,z) = F(x + 2y + 3z - 1, x^3 + y^2 - z^2).$$

(a) Note that $G(-2,3,-1) = F(0,0) = 0$. Show that one can solve the equation $G(x,y,z)$ for z, say $z = g(x,y)$, for (x,y) in a neighborhood B of $(-2,3)$ such that $g(-2,-3) = -1$.

(b) Find $Dg(-2,3)$.

(c) If $D_1D_1F = 3$, $D_1D_2F = -1$, and $D_2D_2F = 5$ at $(0,0)$, find $D_2D_1g(-2,3)$.

Solution:

(a) Let $H(x,y,z) = (x + 2y + 3z - 1, x^3 + y^2 - z^2)$, Then $G = F \circ H$ and, by the chain rule, $DG(x,y,z) = DF(H(x,y,z))DH(x,y,z)$ so that $DG(-2,3,-1) = DF(0,0)DH(-2,3,-1)$. Since

$$DH(x,y,z) = \begin{bmatrix} 1 & 2 & 3 \\ 3x^2 & 2y & -2z \end{bmatrix}$$

one gets

$$DH(p) = \begin{bmatrix} 1 & 2 & 3 \\ 12 & 6 & 2 \end{bmatrix} \quad \text{and} \quad DG(p) = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 12 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 38 & 22 & 12 \end{bmatrix}$$

where $p = (-2,3,-1)$. Since $D_3G(-2,3,-1) = 12 \neq 0$, from the implicit function theorem there exist an open neighborhood B of the point $(-2,3)$ and a unique function $g : B \to \mathbb{R}$ of class C^2 such that $g(-2,3) = -1$ and $G(x,y,g(x,y)) = 0$.

(b) Let $h(x,y) = (x,y,g(x,y))$ for $(x,y) \in B$. Then h is of class C^2 and $(G \circ h)(x,y) = 0$ for $(x,y) \in B$. Abbreviating $q = (x,y)$ and $p = h(q)$, one finds using the chain rule

$$\begin{aligned} \begin{bmatrix} 0 & 0 \end{bmatrix} = D(G \circ h)(q) = DG(p)Dh(q) \begin{bmatrix} D_1G(p) & D_2G(p) & D_3G(p) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ D_1g(q) & D_2g(q) \end{bmatrix} \\ = \begin{bmatrix} D_1G(p) + D_3G(p)D_1g(q) & D_2G(p) + D_3G(p)D_2g(q) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} D_1G(p) & D_2G(p) \end{bmatrix} + D_3G(p) \begin{bmatrix} D_1g(q) & D_2g(q) \end{bmatrix}$$

implying that when $q = (-2,3)$ and $p = h(q) = (-2,3,-1)$, then

$$\begin{aligned} Dg(-2,3) &= \begin{bmatrix} D_1g(-2,3) & D_2g(-2,3) \end{bmatrix} \\ &= -\frac{1}{12} \begin{bmatrix} D_1G(-2,3,-1) & D_2G(-2,3,-1) \end{bmatrix} = -\frac{1}{12} \begin{bmatrix} 38 & 22 \end{bmatrix}. \end{aligned}$$