MATH 404 ANALYSIS II - SPRING 2009  
SOLUTIONS to HOMEWORK 3

Problem 1. Show that if \( U \subset \mathbb{R}^n \) and \( f : U \to \mathbb{R} \), and if the partial derivatives \( D_j f \) exist and are bounded on \( B_z(a) \subset U \), then \( f \) is continuous at \( a \).

Solution: Assume \( |D_j f(x)| < C \) for all \( x \in B_z(a) \) and \( 1 \leq j \leq n \). Let \( h \in \mathbb{R}^n \) be such that \( |h| < \varepsilon \). Define \( k_0 = 0 \) and \( k_j = \sum_{i=1}^{j} h_j e_j \) for \( 1 \leq j \leq n \). Then \( a + k_j \in B_z(a) \). Then

\[
|f(a + h) - f(a)| = \left| \sum_{j=1}^{n} (f(a + k_j) - f(a + k_{j-1})) \right| \leq \sum_{j=1}^{n} |f(a + k_{j-1} + h_j e_j) - f(a + k_{j-1})|.
\]

If \( h_j = 0 \), then \( f(a + k_{j-1} + h_j e_j) - f(a + k_{j-1}) = 0 \). For any \( 1 \leq j \leq n \) such that \( h_j \neq 0 \), let \( g_j(t) = f(a + k_{j-1} + h_j e_j) - f(a + k_{j-1}) \) for \( t \) between 0 and \( h_j \). Since \( D_j \) exists on \( B_z(a) \), \( g_j \) is continuous on \([0, h_j]\) if \( h_j > 0 \) (or on \([h_j, 0]\) if \( h_j < 0 \)) differentiable on \((0, h_j)\) if \( h_j > 0 \) (or on \((h_j, 0)\) if \( h_j < 0 \)). By the mean value theorem, there is \( t_j \) between 0 and \( h_j \) so that \( g_j(h_j) = g_j(t_j) = D_j(a + k_{j-1} + t_j e_j) h_j \). Abbreviating \( \alpha_j = a + k_{j-1} + t_j e_j \), the right hand side of (1) is equal to

\[
\sum_{j=1}^{n} |D_j(a_j^*) h_j| = \sum_{j=1}^{n} |D_j(a_j^*)| \cdot |h_j| \leq C \sum_{j=1}^{n} |h_j| \leq C |h|.
\]

Consequently, \( f \) is continuous at \( a \).

Problem 2. Let \( f : \mathbb{R}^3 \to \mathbb{R} \) and \( g : \mathbb{R}^2 \to \mathbb{R} \) be differentiable. Let

\[
F : \mathbb{R}^2 \to \mathbb{R}, \quad F(x, y) = f(x, y, g(x, y)).
\]

(a) Find \( DF \) in terms of the partial derivatives of \( f \) and \( g \).

(b) If \( F(x, y) = 0 \) for all \((x, y)\), find \( D_1 g \) and \( D_2 g \) in terms of partial derivatives of \( f \).

Solution: Denote by \( G : \mathbb{R}^2 \to \mathbb{R}^3 \) the map defined by \( G(x, y) = (x, y, g(x, y)) \). Then \( G \) is differentiable at each point \((x, y) \in \mathbb{R}^2 \) (a) Note that \( F = f \circ G : \mathbb{R}^2 \to \mathbb{R} \).

By the chain rule, \( DF(x, y) = Df(G(x, y))DG(x, y) \). That is, if \( p = (x, y) \) and \( q = G(p) = (x, y, g(x, y)) \), then

\[
DF(p) = \begin{bmatrix} D_1 F(p) & D_2 F(p) \end{bmatrix} = \begin{bmatrix} D_1 f(q) & D_2 f(q) & D_3 f(q) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ D_1 g(p) & D_2 g(p) \end{bmatrix}
\]

\[
= \begin{bmatrix} D_1 f(q) + D_3 f(q) D_1 g(p) \\ D_2 f(q) + D_3 f(q) D_2 g(p) \end{bmatrix}.
\]
(b) If \( F(x, y) = 0 \) for all \((x, y)\), then \( DF(x, y) = [0 \ 0] \), and, in view of the above identity,

\[
0 = D_1 f(q) + D_3 f(q) D_1 g(p) \quad \text{and} \quad 0 = D_2 f(q) + D_3 f(q) D_2 g(p).
\]

Hence

\[
D_1 g(x, y) = -\frac{D_1 f(x, y, g(x, y))}{D_3 f(x, y, g(x, y))} \quad \text{and} \quad D_2 g(x, y) = -\frac{D_2 f(x, y, g(x, y))}{D_3 f(x, y, g(x, y))}.
\]

**Problem 3.**

(a) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
  x^2 \sin(1/x) & \text{if } x \neq 0, \\
  0 & \text{if } x = 0.
\end{cases}
\]

Show that \( f \) is differentiable at 0 but \( f' \) is not continuous at 0.

(b) Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
g(x, y) = \begin{cases} 
  (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) & \text{if } (x, y) \neq 0, \\
  0 & \text{if } (x, y) = 0.
\end{cases}
\]

Show that \( g \) is differentiable at (0, 0) but \( D_1 g \) is not continuous at (0, 0).

**Solution:** (a) For \( x \neq 0 \), \( f'(x) = 2x \sin(1/x) - \cos(1/x) \) and at \( x = 0 \),

\[
f'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0.
\]

For \( x_j = 1/(2\pi j) \) for \( j \geq 1 \), we have \( x_j \to 0 \) and \( f'(x_j) = -\cos(2\pi j) = -1 \). Hence \( f'(x_j) \neq f'(0) \) so that \( f' \) is not continuous.

(b) We have

\[
D_1 g(0, 0) = \lim_{t \to 0} \frac{g(t, 0)}{t} = \lim_{t \to 0} t \sin(1/t) = 0
\]

\[
D_2 g(0, 0) = \lim_{t \to 0} \frac{g(0, t)}{t} = \lim_{t \to 0} t \sin(1/t) = 0
\]

and for \((x, y) \neq (0, 0)\),

\[
D_1 g(x, y) = 2x \sin(1/\sqrt{x^2 + y^2}) - \frac{x}{\sqrt{x^2 + y^2}} \cos(1/\sqrt{x^2 + y^2})
\]

\[
D_2 g(x, y) = 2y \sin(1/\sqrt{x^2 + y^2}) - \frac{y}{\sqrt{x^2 + y^2}} \cos(1/\sqrt{x^2 + y^2}).
\]

Take \((x_j, y_j) = (1/(2\sqrt{2}\pi j), 1/(2\sqrt{2}\pi j))\). Then \((x_j, y_j) \to (0, 0)\) and \(\sqrt{x_j^2 + y_j^2} = 1/(2\pi j)\). So,

\[
D_1 f(x_j, y_j) = -\frac{1}{\sqrt{2}} \cos(2\pi j) = -\frac{1}{\sqrt{2}}
\]

\[
D_2 f(x_j, y_j) = -\frac{1}{\sqrt{2}} \cos(2\pi j) = -\frac{1}{\sqrt{2}}.
\]

Consequently, \( D_1 f \) and \( D_2 f \) are not continuous at \((0, 0)\). On the other hand, \( g \) is differentiable at \((0, 0)\), then \( Dg(0, 0) = [0 \ 0] \). We calculate the limit

\[
\lim_{(x, y) \to (0, 0)} \frac{g(x, y) - g(0, 0) - 0 \cdot x - 0 \cdot y}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \to (0, 0)} \sqrt{x^2 + y^2} \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) = 0,
\]

so \( Dg(0, 0) = [0 \ 0] \).
showing that \( g \) is differentiable at \((0,0)\).

**Problem 4.** Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
x^2 \arctan\left(\frac{x}{y}\right) - y^2 \arctan\left(\frac{x}{y}\right) & \text{if } xy \neq 0, \\
0 & \text{if } xy = 0. 
\end{cases}
\]
Show that \( D_1 D_2 f(0) \neq D_2 D_1 f(0) \).

**Solution:**
\[
D_1 f(0, y) = \lim_{x \to 0} \frac{f(x, y) - f(0, y)}{x} = \begin{cases} 
\lim_{x \to 0} x \arctan\left(\frac{x}{y}\right) - \frac{x^2}{y^2} \arctan\left(\frac{x}{y}\right) & y \neq 0 \\
0 & y = 0
\end{cases} = \begin{cases} 
-1 & y \neq 0 \\
0 & y = 0
\end{cases}
\]
and similarly,
\[
D_2 f(x, 0) = \begin{cases} 
1 & x \neq 0 \\
0 & x = 0.
\end{cases}
\]
Hence
\[
D_2 D_1 f(0, 0) = \lim_{y \to 0} \frac{D_1 f(0, y) - D_1 f(0, 0)}{y} = -1
\]
\[
D_1 D_2 f(0, 0) = \lim_{x \to 0} \frac{D_2 f(x, 0) - D_2 f(0, 0)}{y} = 1.
\]

**Problem 5.** Let \( g : \mathbb{R} \to \mathbb{R} \) be of class \( C^2 \). Show that
\[
\lim_{h \to 0} \frac{g(a + h) - 2g(a) + g(a - h)}{h^2} = g''(a).
\]

**Hint:** Let \( f(x, y) = g(x + y) \). If \( h > 0 \) is small, let \( D(h) = [f(a/2 + h/2, a/2 + h/2) - f(a/2 - h/2, a/2 + h/2)] - [f(a/2 + h/2, a/2 - h/2) - f(a/2 - h/2, a/2 - h/2)] \). Argue like in the proof of the theorem about mixed derivatives to show that \( D(h)/h^2 \to D_2 D_1 f(a/2, a/2) \).

**Solution:** Let \( f(x, y) = g(x + y) \). Since \((x, y) \mapsto x + y\) is of class \( C^\infty \) and \( g \) is of class \( C^2 \), the map \( f \) is of class \( C^2 \).
\[
D(h) := [f(a/2 + h/2, a/2 + h/2) - f(a/2 - h/2, a/2 + h/2)] - [f(a/2 + h/2, a/2 - h/2) - f(a/2 - h/2, a/2 - h/2)]
\]
and
\[
r(h) := f(a/2 + h/2, a/2 + t) - f(a/2 - h/2, a/2 + t).
\]
So,
\[
D(h) = r(h/2) - r(-h/2).
\]
The function \( g \) is continuous on \([-h/2, h/2]\) and is differentiable on \((-h/2, h/2)\). So, applying the mean value theorem, there is \( s^* \in (-h/2, h/2) \) so that
\[
(2) \quad D(h) = r'(s^*)h = [D_2 f(a/2 + h/2, a/2 + s^*) - D_2 f(a/2 - h/2, a/2 + s^*)]h.
\]
The function \( t \mapsto D_2 f(a/2 + t, a/2 + s^*) \) is continuous on \((-h/2, h/2]\) and differentiable on \([-h/2, h/2)\). So, applying again the mean value theorem, there is \( t^* \in (-h/2, h/2) \) for which the right side of (2) is equal to
\[
D_1 D_2 f(a/2 + t^*, a/2 + s^*)h^2.
\]
Hence $D(h) = D_1 D_2 f(a/2 + t*, a/2 + s*) h^2$. On the other hand, $D(h) = g(a + h) - 2g(a) + g(a - h)$ and $D_1 D_2 f(a/2 + t*, a/2 + s*) = g''(a + s^* + t^*)$. Moreover, if $h \to 0$, then $s^*, t^* \to 0$ so that

$$
\frac{g(a + h) - 2g(a) + g(a - h)}{h^2} = \frac{D(h)}{h^2} = D_1 D_2 f(a/2 + t^*, a/2 + s^*)
$$

$$
= g''(a + s^* + t^*) \to g''(a)
$$

since $g''$ is continuous.