MATH 404 ANALYSIS - SPRING 2008

SOLUTIONS to HOMEWORK 3

1. Consider the subset A of $\mathbb{R}^2$ defined by $A = (-1,1) \times \{0\}$ and let U be the open ball $U = B_1(0)$ in $\mathbb{R}^2$. Show that there is no $\varepsilon > 0$ such that the ε-neighborhood of A is contained in U.

Solution:
Assume that there is $\varepsilon > 0$ such that $U(A,\varepsilon) = \bigcup_{a\in A} B_\varepsilon(a) \subset B_1(0)$. Take the point $a = (1-(\varepsilon/2), 0) \in A$ and let $x = (1,0)$. Then $x \notin B_1(0)$ but $\|x-a\| = \sqrt{(1-(1-\varepsilon/2))^2} = \varepsilon/2 < \varepsilon$. Hence $x \in B_\varepsilon(a) \subset U(A,\varepsilon) \subset B_1(0)$, contradiction.

2. Let $\mathbb{R}^\infty$ be the set of all "infinite-tuples" $x = (x_1, x_2, \ldots)$ of real numbers that end in an infinite string of 0's. Define an inner product on $\mathbb{R}^\infty$ by $\langle x, y\rangle = \sum_{i=1}^\infty x_i y_i$ (this is a finite sum since all but finitely many terms vanish) and let $d(x,y) = \|x-y\| := \sqrt{\langle x, x\rangle}$ be the metric on $\mathbb{R}^\infty$. Define

$$e_i = (0,\ldots,0,1,0,\ldots,0,\ldots),$$

where 1 appears in the ith place. Then the e_i's form a basis for $\mathbb{R}^\infty$. Let

$$A = \{e_i \mid i \geq 1\}.$$

Show that A is closed, bounded, and non-compact.

Solution: Clearly, the set A is bounded since $\|e_i\| = 1$ for all $i \geq 1$. To see that A is closed, first note that

$$\|e_i - e_j\|^2 = \begin{cases} 2 & i \neq j \\ 0 & i = j. \end{cases} \qquad (1)$$

Let $(x^n) = (e_{i_n})$ be such that $\|x^n - x\| \to 0$ for some $x \in \mathbb{R}^\infty$. We have to show that $x = e_i$ for some i. Since (x^n) converges, it is Cauchy, that is, $\|x^n - x^m\| \to 0$ as $n, m \to \infty$. Hence by (1) we must have $e_{i_n} = e_{i_m}$ for all $n, m \geq N$. Then $x = e_i$ with $i = I_N$.
To see that A is not compact, take teh sequence $(x^n) = (e_n)$. Then we have $\|x^n - x^m\| = \sqrt{2}$ for all $n \neq m$. Such a sequence can't have a convergent subsequence. Hence A is not compact.

3. Let $A \subset \mathbb{R}^m$ and let $f : A \to \mathbb{R}^n$. Show that if $f'(a;u)$ exists, then $f'(a;ru)$ exists and $f'(a;ru) = rf'(a;u)$.

Solution:
Take $u \in \mathbb{R}^n \setminus \{0\}$ and $r \in \mathbb{R}$. Then

$$\frac{f(a+tru)-f(a)}{t} = r\frac{f(a+(tr)u)-f(a)}{tr} = r\frac{f(a+su)-f(a)}{s}$$

where $s = tr$. If $t \to 0$, then $s = tr \to 0$ and since $f'(a,u)$ exists the right-hand side converges to $rf'(a,u)$ so that the limit on the left converges. Hence $f'(a,ru) = rf'(a,u)$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0), \\ \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0). \end{cases}$$

(a) For which vectors $u \neq (0,0)$ does $f'((0,0);u)$ exist? Evaluate it when it exist.
(b) Do $D_1 f$ and $D_2 f$ exist at $(0,0)$.
(c) Is f differentiable at $(0,0)$?
(d) Is f continuous at $(0,0)$.

Solution:
(a) Let $u = (u_1, u_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then

$$\frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} = \frac{f(tu_1,tu_2)}{t} = \frac{1}{t}\frac{u_1 u_2}{u_1^2+u_2^2}.$$

If $u_1 \neq 0$ and $u_2 \neq 0$, then the right hand side converges to ∞ as $t \in 0$. However, if $u_1 = 0$ (and then $u_2 \neq 0$) or $u_2 = 0$ (and then $u_1 \neq 0$), then the right hand side is equal to 0. Hence if $u = (u_1,u_2) \neq (0,0)$, then

$$f'((0,0);(u_1,u_2)) = \begin{cases} 0 & u_1 = 0 \text{ or } u_2 = 0 \\ \text{doesn't exist} & u_1 \neq 0 \text{ and } u_2 \neq 0. \end{cases}$$

(b) By (a), $D_1 f(0,0) == D_2 f(0,0) = 0$.
(c) and (d) f is not continuous at $(0,0)$ since $(1.n.1/n) \to (0,0)$ but $f(1/n,1/n) = 1/2 \not\to 0 = f(0,0)$. Since f is not continuous at $(0,0)$, it is not differentiable at $(0,0)$.

5. Repeat Problem 4 for the function

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0), \\ \dfrac{x\,|y|}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0). \end{cases}$$

Solution:
(a) Let $u = (u_1,u_2) \neq (0,0)$. Then

$$\begin{aligned} \frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} &= \frac{f(tu_1,tu_2)}{t} = \frac{1}{t}\frac{t\,|t|\,u_1\,|u_2|}{\sqrt{t^2(u_1^2+u_2^2)}} \\ &= \frac{t\,|t|\,u_1\,|u_2|}{\sqrt{t^2(u_1^2+u_2^2)}} = \frac{t\,|t|\,u_1\,|u_2|}{t\,|t|\,\sqrt{u_1^2+u_2^2}} = \frac{u_1\,|u_2|}{\sqrt{u_1^2+u_2^2}}. \end{aligned}$$

Hence $'((0,0);(u_1,u_2))$ exists for every $u = (u_1,u_2) \neq (0,0)$ and $f'((0,0);(u_1,u_2)) = \dfrac{u_1\,|u_2|}{\sqrt{u_1^2+u_2^2}}$.
(b) By (a), $D_1 f(0,0) = D_2 f(0,0) = 0$.
(c) Note that

$$\frac{1}{\sqrt{x^2+y^2}}[f(x,y) - f(0,0) - D_1 f(0,0)x - D_2 f(0,0)y] = \frac{x\,|y|}{x^2+y^2}$$

Since $\dfrac{t\,|t|}{t^2+t^2} = |t|\,/2t$ doesn't converge to 0 as $t \to 0$, f is not differentiable at $(0,0)$

(d) The function f is continuous at $(0,0)$. Indeed,

$$|f(x,y)| = \frac{|xy|}{x^2+y^2} = \sqrt{|xy|} \cdot \sqrt{\frac{|xy|}{x^2+y^2}} \leq \sqrt{|xy|}/\sqrt{2}$$

since $\dfrac{|xy|}{x^2+y^2} \leq 1/2$. Since $|xy| \leq (x^2+y^2)/2 \to 0$ as $(x,y) \to 0$, we have $|f(x,y)| \to 0$ as $(x,y) \to (0,0)$.