MATH 404 ANALYSIS - SPRING 2008

SOLUTIONS to HOMEWORK 2

1. Let K be a subset of $\mathbb{R}^n$. Show that if every continuous function $f : K \to \mathbb{R}$ is bounded, then K is compact.

Solution:
We show that if K is not compact, then there exists a continuous function which is unbounded. If K is not closed, then there exists a point $x_0 \in \overline{K}$ such that $x_0 \notin K$. Consider $f(x) = 1/\|x_0 - x\|$ for $x \in K$. Then f is continuous and if $(x_n) \subset K$ converges to x_0, then $\|x_0 - x_n\| \to 0$ showing that $f(x_n) \to \infty$. If K is unbounded, then consider $f(x) = \|x\|$ for $x \in K$. Then f is continuous and since K is unbounded, there is a sequence $(x_n) \subset K$ such that $\|x_n\| \to \infty$. Hence $f(x_n) \to \infty$.

2. Let $A \subset \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be uniformly continuous.

(a) Show that the sequence $(f(x_k))$ is Cauchy in $\mathbb{R}^m$ whenever (x_k) is Cauchy in A.
(b) Show that there exists a unique continuous function $g : \overline{A} \to \mathbb{R}^m$ such that $g(x) = f(x)$ for all $x \in A$.

Solution:
(a) Let $(x_n) \subset A$ be Cauchy and let $\varepsilon > 0$. Since f is uniformly continuous, there is $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ for all $x, y \in A$ satisfying $\|x - y\| < \delta$. Since (x_n) is Cauchy, there is $N \in \mathbb{N}$ such that $\|x_k - x_m\| < \delta$ for $k, m \geq N$. Hence $\|f(x_k) - f(x_m)\| < \varepsilon$ for $n, m \geq N$, i.e., $(f(x_k))$ is Cauchy.

(b) If $x \in \overline{A}$, then there is $(x_n) \in A$ such that $x_n \to x$. Since we want the g to be continuous, we must have $g(x) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} f(x_k)$. We must show that this definition of $g(x)$ does not depend on the choice of the sequence converging to x. Assume that (y_k) is another sequence in A converging to x. Then the sequence $x_1, y_1, x_2, y_2, \ldots)$ also converges to x. Hence it is Cauchy and so, the sequence $(f(x_1), f(y_1), f(x_2), f(y_2), \ldots)$ is Cauchy by (a). So, it converges. Hence every subsequence converges to the same limit. Hence $(f(x_k))$ and $(f(y_k))$ converge to the same limit showing that the function g is well-defined. To see that there is exactly one function we argue by contradiction and assume that $h : \overline{A} \to \mathbb{R}^m$ is another continuous function satisfying $h(x) = f(x)$ for $x \in A$. For the function $F = g - h$ on $\overline{A}$ we have $F(x) = 0$ for $x \in A$. Hence $F = 0$ on $\overline{A}$ since F is continuous. So $g = h$. Finally, to see that g is continuous take $\varepsilon > 0$ and let $\delta > 0$ be such that $\|f(x) - f(y)\| < \varepsilon$ for all $x, y \in A$ satisfying $\|x - y\| < \delta$. Take $x_0, y_0 \in \overline{A}$ such that $\|x_0 - y_0\| < \delta/2$. Then there are sequences $(x_n), (y_n) \subset A$ such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Then $\|x_n - y_n\| < \delta$ for large n. Hence $\|f(x_n) - f(y_n)\| < \varepsilon$ for large n. Taking the limit $n \to \infty$, we get $\|g(x_0) - g(y_0)\| = \lim \|f(x_n) - g(x_n)\| \leq \varepsilon$. Hence g is continuous (in fact uniformly continuous) on $\overline{A}$.

3. Let $A \subset \mathbb{R}^n$.

(a) Show that $\overline{A} = \{x \in \mathbb{R}^n \,|\, d(x, A) = 0\}$. Conclude that $d(x, A) > 0$ if A is closed and $x \notin A$.

(b) If A is closed, show that $A = \bigcap_{k=1}^{\infty} O_k$ where each set O_k is open in $\mathbb{R}^n$. *Hint*: Use $f(x) = d(x, A)$.

Solution:

(a) Take $x \in \overline{A}$. Then $\|x - x_n\| \to 0$ for some sequence $(x_n) \subset A$. Since $d(x, A) \le \|x - x_n\| \to 0$, $d(x, A) = 0$. This shows that $\overline{A} \subset \{x \in \mathbb{R}^n |\, d(x, A) = 0\}$. Conversely, if $d(x, A)$, then by the definition fo $d(x, A)$ for every n, there is $x_n \in A$ such that $\|x_n - x\| \le d(x, A) + 1/n = 1/n$. Hence $x_n \to x$ and $x \in \overline{A}$.

(b) Consider the function $f : \mathbb{R}^n \to [0, \infty)$ defined by $f(x) = d(x, A)$. For every $k \in \mathbb{N}$, define $O_k := f^{-1}([0, 1/k)) = \{x \in \mathbb{R}^n |\, 0 \le f(x) < 1/k\}$. Since f is continuous and $[0, 1/k)$ is open in $[0, \infty)$, every set O_k is open. Clearly, $\overline{A} \subset O_k$ since $\overline{A} = \{x \in \mathbb{R}^n |\, d(x, A) = 0\}$ by (a). So, $\overline{A} \subset \bigcap_{k \ge 1} O_k$. Conversely, if $x \in \bigcap_{k \ge 1} O_k$, then $0 \le f(x) < 1/k$ for all k. So, $f(x) = 0$, i.e., $d(x, A) = 0$ and $x \in \overline{A}$.

4. Let A and B be disjoint closed subsets of $\mathbb{R}^n$. Show that there are open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. *Hint*: Consider the function $f(x) = \dfrac{d(x, A)}{d(x, A) + d(x, B)}$.

Solution:

First note that $d(x, A) + d(x, B) > 0$ for every $x \in \mathbb{R}^n$. Hence the function $f(x) = \dfrac{d(x, A)}{d(x, A) + d(x, B)}$, $x \in \mathbb{R}^n$, is well-defined. It is also continuous. and takes values in $[0, 1]$. Take $U = f^{-1}([0, 1/3)) = \{x \in \mathbb{R}^n |\, 0 \le f(x) < 1/3\}$ and $V = f^{-1}([0, 1/3)) = \{x \in \mathbb{R}^n |\, 2/3 < f(x) \le\}$. Then U and V are disjoint, open, and $A \subset U$, $B \subset V$.

5. Let $K \subset \mathbb{R}^n$ be compact and let $f : K \to K$ be such that $\|f(x) - f(y)\| = \|x - y\|$. Show that f is surjective. *Hint*: Argue by contradiction and assume that there is $x_0 \in K \setminus f(K)$. Set $\alpha = d(x_0, f(K))$. Then $\alpha > 0$ (why?). Define the sequence $(x_k) \subset f(K)$ by $x_{k+1} = f(x_k)$ for $k \ge 0$. Show that $\|x_k - x_m\| \ge \alpha$ for all $k, m \in \mathbb{N}$. Reach a contradiction.

Solution:

Arguing by contradiction assume that there is $x_0 \in K \setminus f(K)$. The function f is continuous and the set K is compact. Hence $f(K)$ is compact and since $x_0 \notin f(K)$, $\alpha = d(x_0, f(K)) > 0$. Define $x_{k+1} = f(x_k)$ for $k \ge 0$. Since $(x_k) \subset K$ and K is compact, the sequence (x_n) has a converging subsequence. On the other hand, if $k > m$, then by the assumption

$$
\begin{aligned}
\|x_k - x_m\| &= \|f(x_{k-1}) - f(x_{m-1})\| = \|x_{k-1} - x_{m-1}\| \\
&= \|f(x_{k-2}) - (x_{m-2})\| = \cdots = \|f(x_{k-m}) - f(x_0\| \\
&= \|x_{k-m} - x_0\| = \|f(x_{k-m-1}) - x_0\| \ge \alpha
\end{aligned}
$$

since $f(x_{k-m-1}) \in f(K)$. Since $\|x_k - x_m\| \ge \alpha$, the sequence (x_k) doesn't have a converging subsequence, contradiction.