Problem 1. Let $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ and let $g : S^1 \to \mathbb{R}$ be a continuous function satisfying $g(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} |x| \cdot g\left(\frac{x}{|x|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show that if $x \in \mathbb{R}^2$ and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = f(tx)$ then $h$ is differentiable on $\mathbb{R}$.

(b) Assume that $\lim_{\substack{\|h\| \\ (h,k) \to (0,0)}} |f(h,k) - Ah - Bk| = 0$. If $f$ were differentiable at $(0,0)$, then $Df(0,0) = [A B]$ and

$$\lim_{(h,k) \to (0,0)} \frac{|f(h,k) - Ah - Bk|}{\sqrt{h^2 + k^2}} = 0.$$

Consider $(0,k)$ with $k \neq 0$. Then $h(0,k) = \|k\| g((0,0)/\|k\|) = 0$ so that

$$\lim_{(0,k) \to (0,0)} \frac{|f(0,0) - Bk|}{\sqrt{k^2}} = \lim_{k \to 0} \frac{|Bk|}{\|k\|} = |B| = 0.$$

Similarly, considering $(h,0)$ one finds that $A = 0$. Since $g \neq 0$, then there is $(x,y) \in S^1$ such that $g(x,y) \neq 0$. Hence $|(x,y)| = 1$ and if $(h,k) = (tx,ty)$ then $f(tx,ty) = tg((x,y))$ so that

$$\lim_{(h,k) \to (0,0)} \frac{|f(h,k) - Ah - Bk|}{\sqrt{h^2 + k^2}} = \lim_{t \to 0} \frac{|f(tx,ty)|}{|t|} = |g(x,y)| \neq 0,$$

contradiction.

(c) Take $g(x,y) = x|y|$ for $(x,y) \in S^1$. Then $g$ is continuous and satisfy $g(1,0) = g(0,1) = 0$ and $g(-(x,y)) = -g((x,y))$. Then for $(x,y) \neq (0,0),

$$\sqrt{x^2 + y^2} g\left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right) = \sqrt{x^2 + y^2} \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right) = \frac{x|y|}{\sqrt{x^2 + y^2}} = f(x,y).$$
Problem 2. Let $\langle \cdot , \cdot \rangle$ be an inner product on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be defined by $f(x, y) = \langle x, y \rangle$.

(a) Show that $f$ is differentiable at any point $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ and find its derivative $Df(a, b)$.

(b) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}^n$ be differentiable functions and let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$h(t) = \langle f(t), g(t) \rangle, \quad t \in \mathbb{R}.$$

Show that

$$h'(t) = \langle Df(a)^t, g(t) \rangle + \langle f(t), Dg(a)^t \rangle.$$

Note that $Df(a)$ is an $n$ by 1 matrix, its transpose $Df(a)^t$ is a 1 by $n$ matrix which we consider as an element of $\mathbb{R}^n$.

Solution: (a)

$$f(a + h, b + k) - f(a, b) - (a + h, b + k) - (a, b) = \langle a, b \rangle + \langle a, k \rangle + \langle h, b \rangle + \langle h, k \rangle - \langle a, b \rangle$$

$$= \langle a, k \rangle + (b, h) + \langle h, k \rangle.$$

Note that $(h, k) \mapsto \langle a, k \rangle + (b, h)$ is linear and

$$\langle h, k \rangle \leq |h| \cdot |k| \leq \frac{1}{2}(|h|^2 + |k|^2) = \frac{1}{2}|(h, k)|^2.$$

Hence

$$\frac{|f(a + h, b + k) - f(a, b) - (a + h, b + k) - (a, b)|}{|(h, k)|} \leq \frac{1}{2} \cdot \frac{|(h, k)|^2}{|(h, k)|} = \frac{1}{2} |(h, k)| \rightarrow 0$$

as $(h, k) \rightarrow (0, 0)$. Hence $f$ is differentiable with $Df(a)(h, k) = \langle a, k \rangle + (b, h)$.

(b) Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ be defined by $\alpha(t) = (f(t), g(t))$ and let $\beta(x, y) = \langle x, y \rangle$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then $\alpha$ is differentiable at any $t$ with $D\alpha(t) = (Df(t), Dg(t))$ (this is a 2n by 1 column). Since $h(t) = \beta \circ \alpha(t)$ and $\alpha$ and $\beta$ are differentiable, it follows from part (a) and the chain rule that

$$h'(t) = Dh(a) = D\beta \circ \alpha(a) = \langle f(a), Dg(a)^t \rangle + \langle g(a), Df(a)^t \rangle$$

$$= \langle f(a), Dg(a)^t \rangle + \langle Df(a)^t, g(a) \rangle.$$

Problem 3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$f(x) = \sum_{i=1}^{n} |x_i|.$$

Determine the set of points where $f$ is differentiable.

Solution: Consider $f(x) = \sum_{i=1}^{n} |x_i|$. Claim: the function $f$ is differentiable only at the points $x = (x_1, \ldots, x_n)$ such that $x_i \neq 0$ for all $1 \leq i \leq n$. Indeed, assume $x = (x_1, \ldots, x_n)$ with $x_i \neq 0$ for all $1 \leq i \leq n$. Fix $1 \leq j \leq n$. Then for $t$ small,

$$\frac{f(x + te_j) - f(x)}{t} = \frac{|x_j + t| - |x_j|}{t}.$$

Since $x_j \neq 0$, then for $t$ small $|x_j + t| - |x_j| = t$ if $x_j > 0$ and $|x_j + t| - |x_j| = -t$ if $x_j < 0$. Hence the partial derivative $D_j f(x)$ exists and is equal to 1 when $x_j > 0$ and to $-1$ when $x_j < 0$. Since the partial derivatives are constant on a small open
neighborhood of \( x \), hence continuous, it follows that \( f \) is differentiable at such a point \( x \). If \( x = (x_1, \ldots, x_n) \) with \( x_j = 0 \), then

\[
\lim_{t \to 0} \frac{f(x + te_j) - f(x)}{t} = |t|
\]

The limit of the above quotient does not exist as \( t \to 0 \). Hence \( D_j f(x) \) doesn’t exist and \( f \) is not differentiable at such point \( x \).

4. Let \( f : \mathbb{R}^n \to \mathbb{R} \). If \( v \in \mathbb{R}^n \), then the limit

\[
\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}
\]

if exists, is denoted by \( D_v f(a) \), and is called the directional derivative of \( f \) at \( a \) in the direction \( v \).

(a) Show that \( D_v f(a) = D_v f(a) \) and \( D_{av} f(a) = \alpha D_v f(a) \).

(b) If \( f \) is differentiable at \( a \), show that \( D_v f(a) = D f(a) v \) and therefore \( D_{av} f(a) = D_u f(a) + D_v f(a) \).

(c) Let \( f \) be the function defined in Problem 1. Show that \( D_v f(0, 0) \) exists for all \( v \) but unless \( g = 0 \), then \( D_{u+v} f(0, 0) = D_u f(0, 0) + D_v f(0, 0) \) is not true for all \( u \) and \( v \).

Solution: (a) The equality \( D_v f(a) = D_v f(a) \) follows from the definitions. To see that \( D_{av} f(a) = \alpha D_v f(a) \), one calculates for \( \alpha \neq 0 \)

\[
D_{av} f(a) = \lim_{t \to 0} \frac{f(a + t\alpha v) - f(a)}{t} = \alpha \lim_{t \to 0} \frac{f(a + t\alpha v) - f(a)}{\alpha t} = \alpha \frac{f(a + sv) - f(a)}{s} = \alpha D_v f(a).
\]

(b) It suffices to show that \( \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} \) exists and is equal to \( D f(a) v \). To see this, calculate

\[
\begin{align*}
\frac{f(a + tv) - f(a)}{t} - D f(a) v &= \frac{f(a + tv) - f(a) - D f(a) tv}{t} \\
&= |v| \left( \frac{f(a + tv) - f(a) - D f(a) tv}{|tv|} \right) \\
&= |v| \frac{f(a + tv) - f(a) - D f(a) tv}{|tv|}.
\end{align*}
\]

If \( t \to 0 \), then \( tv \to 0 \) so that 

\[
\left( \frac{f(a + tv) - f(a) - D f(a) tv}{|tv|} \right) \to 0
\]

since \( f \) is differentiable at \( a \). Hence

\[
\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} - D f(a) v = 0 \quad \text{as} \quad t \to 0
\]

which implies that

\[
\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = D f(a) v.
\]

Next \( D_{u+v} f(a) = D f(a) (u + v) = D f(a) u + D f(a) v = D_u f(a) + D_v f(a) \).

(c) For \( v = 0 \), \( D_v f(0) = 0 \). So, take \( v \neq 0 \). Then

\[
f(tv) = |t||v| g(\frac{t|v|}{|v|}) = t|v| g(v)
\]
since $g$ is odd. Hence

$$\lim_{t \to 0} \frac{f(tv) - f(0)}{t} = |v| g\left(\frac{v}{|v|}\right).$$

Next assume that $g \neq 0$. Then there is $w = (x, y) \in S^1$ so that $g(x, y) \neq 0$. Set $u = (x, 0)$ and $v = (0, y)$. Then $g(u/|u|) = g(\pm(1,0)) = 0$ so that $D_u f(0, 0) = 0$ and similarly $D_v f(0, 0) = 0$. However, $D_u f(0, 0) = g(w) = g(x, y) \neq 0$ so that the equality $D_{u+v} f(0, 0) = D_u f(0, 0) + D_v f(0, 0)$ does not hold for all $u$ and $v$. 