Problem 1. Let \( f(x, y) = \frac{1}{(y+1)^2} \) and let
\[
A = \{(x, y) \in \mathbb{R}^2 | x > 0 \text{ and } x < y < 2x\},
\]
\[
B = \{(x, y) \in \mathbb{R}^2 | x > 0 \text{ and } x^2 < y < 2x^2\}.
\]
Show: \( \int_A f \) does not exist and \( \int_B f \) exists. Find the value of \( \int_B f \).

Problem 2. Let \( U = \{(x, y, z) \in \mathbb{R}^3 | x > 0 \text{ and } x^2 + y^2 + z^2 < r^2\} \) and let \( f(x, y) = \frac{1}{(x^2 + y^2)^{3/2}} \) for \((x, y) \neq (0, 0)\). Determine if the function \( f \) is integrable over \( U \) and over \( \mathbb{R}^2 \setminus U \).

Problem 3. Let \( \pi_k : \mathbb{R}^n \to \mathbb{R} \) be the projection onto the \( k \)-th factor, i.e., \( \pi_k(x) = x_k \). If \( S \) is a bounded Jordan-measurable subset of \( \mathbb{R}^n \) with nonzero volume, define the centroid \( c(S) \) of \( S \) to be the point in \( \mathbb{R}^n \) whose \( k \)-th coordinate is equal to
\[
c(S)_k := \frac{1}{v(S)} \int_S \pi_k.
\]
(a) \( S \) is said to be symmetric with respect to the subspace \( x_k = 0 \) if \( g(S) = S \) where \( g : \mathbb{R}^n \to \mathbb{R}^n \) is defined by
\[
g(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, -x_k, x_{k+1}, \ldots, x_n).
\]
Show that if \( S \) is symmetric with respect to the subspace \( x_k = 0 \), then \( c(S)_k = 0 \).
(b) Let \( U = \{(x, y, z) \in \mathbb{R}^3 | x > 0 \text{ and } x^2 + y^2 + z^2 < r^2\} \). Use the spherical coordinates and the change of variables theorem to compute \( c(U) \).

Problem 4. Let \( A \) be an bounded open Jordan-measurable subset of \( \mathbb{R}^{n-1} \) and let \( p = (p_1, \ldots, p_n) \in \mathbb{R}^n \) with \( p_n > 0 \). Define the subset \( S \) of \( \mathbb{R}^n \) by setting \( S = \{x \in \mathbb{R}^n | x = (1-t)a + tp \text{ where } t \in (0,1) \text{ and } a \in A \times \{0\}\} \).
(a) Define a diffeomorphism \( g \) between \( S \) and \( A \times (0,1) \).
(b) Calculate the volume \( v(S) \) in terms of the volume \( v(A) \) of \( A \).
(c) Express the centroid \( c(S) \) in terms of the centroid of \( c(A) \) and \( p \) and show that \( c(S) \) lies on the line segment connecting \((c(A), 0)\) and \( p \).

Problem 5. Let \( B^n(r) \) be a closed ball in \( \mathbb{R}^n \) of radius \( r \) and centered at 0.
(a) Show that \( v(B^n(r)) = \alpha_n r^n \) where \( \alpha_n = v(B^n(1)) \).
(b) Calculate \( \alpha_1 \) and \( \alpha_2 \).
(c) Compute \( \alpha_n \) in terms of \( \alpha_{n-1} \).
(d) Find the formula for \( \alpha_n \) by considering two cases: \( n \) is odd and \( n \) is even.