MATH 404 ANALYSIS - Spring 2008

HOMEWORK 5– Due Tuesday, March 25

1. Abbreviate by $\mathbb{R}_+ = (0, \infty)$. Define $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by

$$f(x, y) = (x + y, \frac{x}{y})$$

(a) Show that f is one-to-one and $f(\mathbb{R}^2_+) = \mathbb{R}^2_+$.

(b) Find the inverse $g : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ of f and show that f and g are of class C^∞.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $f(x) = \|x\|^2 \cdot x$. Show that f is of class C^∞ and that f carries the unit ball $B_1(0)$ onto itself in a one-to-one fashion. Show that the inverse function of f is not differentiable at 0.

3. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$g(x, y) = (2ye^{2x}, xe^y),$$

and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x, y) = (3x - y^2, 2x + y, xy + y^3).$$

(a) Show that there is a neighborhood of $(0, 1)$ that g carries in a one-to-one fashion onto a neighborhood of $(2, 0)$.

(b) Find $D(f \circ g^{-1})$ at $(2, , 0)$.

4. Let A be open in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^n$ be of class C^r. Assume that $Df(x)$ is non-singular for every $x \in A$. Show that even if f is not one-to-one on A, the set $B = f(A)$ is open in $\mathbb{R}^n$.

5. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be of class C^2 with $F(0, 0) = 0$ and $DF(0, 0) = \begin{bmatrix} 2 & 3 \end{bmatrix}$. Let $G : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$G(x, y, z) = F(x + 2y + 3z - 1, x^3 + y^2 - z^2).$$

(a) Note that $G(-2, 3, -1) = F(0, 0) = 0$. Show that one can solve the equation $G(x, y, z)$ for z, say $z = g(x, y)$, for (x, y) in a neighborhood B of $(-2, 3)$ such that $g(-2, 3) = -1$.

(b) Find $Dg(-2, 3)$.

(c) If $D_1D_1F = 3$, $D_1D_2F = -1$, and $D_2D_2F = 5$ at $(0, 0)$, find $D_2D_1g(-2, 3)$.