Problem 1. Let $A$ be a rectangle in $\mathbb{R}^k$ and let $B$ be a rectangle in $\mathbb{R}^n$. Set $Q = A \times B$ and assume that $f : Q \to \mathbb{R}$ is a bounded function. Show that if $\int_Q f$ exists, then $\int_B f(x, y)$ exists for every $x \in A \setminus D$ where $D$ is a set of measure zero in $\mathbb{R}^k$.

Problem 2. Let $A \subset \mathbb{R}^2$ be open and let $f : A \to \mathbb{R}$ be of class $C^2$. Let $Q$ be a closed rectangle contained in $A$.

(a) Use Fubini’s theorem and the fundamental theorem of calculus to show that
\[
\int_Q D_2 D_1 f = \int_Q D_1 D_2 f.
\]

(b) Use (a) to give a proof that $D_1 D_2 f = D_2 D_1 f$ on $A$.

Problem 3.

(a) Construct a set $C \subset [0, 1] \times [0, 1]$ such that $C$ contains at most one point on each horizontal and each vertical line but $\partial C = [0, 1] \times [0, 1]$. Hint; It suffices to ensure that $C$ contains points in each quarter of the square $[0, 1] \times [0, 1]$ and also in each sixteenth, etc.

(b) Let $C$ be the set from (a). Show that
\[
\int_{[0, 1]} \left[ \int_{[0, 1]} \chi_C(x, y) \, dx \right] dy = \int_{[0, 1]} \left[ \int_{[0, 1]} \chi_C(x, y) \, dy \right] dx
\]

but $\int_{[0, 1] \times [0, 1]} \chi_C$ does not exist.

Problem 4. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous and assume that $D + 2f$ is continuous. Define $F(y) := \int_c^b f(x, y) \, dx$ for $y \in [c, d]$. Prove the Leibnitz’s rule:
\[
F'(y) = \int_a^b D_2 f(x, y) \, dx.
\]

Hint: $F(y) = \int_a^b f(x, y) \, dx = \int_a^b \left[ \int_c^y D_2 f(x, y) \, dy + f(x, c) \right] \, dx$.

Problem 5. Let $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Prove that $f$ is of class $C^\infty$. To do this define for every $n \geq 0$ define $f_n : \mathbb{R} \to \mathbb{R}$ by
\[
f_n(x) = \begin{cases} 
\frac{e^{-1/x}}{x^n} & x > 0 \\
0 & x \leq 0.
\end{cases}
\]
(a) Show that $f_n$ is continuous at 0. \textit{Hint:} Show that $a < e^a$ for all $a$. Set $a = t/(2n)$ to conclude that $t^n/e^t < (2n)^n/e^{t/2}$. Then set $t = 1/x$ and let $x \to 0^+$. 

(b) Show that $f$ is differentiable at 0.

(c) Show that $f'_n(x) = f_{n+1}(x) - nf_{n+1}(x)$.

(d) Show that $f_n$ is of class $C^\infty$. 