MATH 404 ANALYSIS - Spring 2008

HOMEWORK 2– Due Thursday, February 14

1. Let K be a subset of $\mathbb{R}^n$. Show that if every continuous function $f : K \to \mathbb{R}$ is bounded, then K is compact.

2. Let $A \subset \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be uniformly continuous.

(a) Show that the sequence $(f(x_k))$ is Cauchy in $\mathbb{R}^m$ whenever (x_k) is Cauchy in A.

(b) Show that there exists a unique continuous function $g : \overline{A} \to \mathbb{R}^m$ such that $g(x) = f(x)$ for all $x \in A$.

3. Let $A \subset \mathbb{R}^n$.

(a) Show that $\overline{A} = \{x \in \mathbb{R}^n | \, d(x, A) = 0\}$. Conclude that $d(x, A) > 0$ if A is closed and $x \notin A$.

(b) If A is closed, show that $A = \bigcap_{k=1}^{\infty} O_k$ where each set O_k is open in $\mathbb{R}^n$. *Hint*: Use $f(x) = d(x, A)$.

4. Let A and B be disjoint closed subsets of $\mathbb{R}^n$. Show that there are open sets U and V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. *Hint*: Consider the function $f(x) = \dfrac{d(x, A)}{d(x, A) + d(x, B)}$.

5. Let $K \subset \mathbb{R}^n$ be compact and let $f : K \to K$ be such that $\|f(x) - f(y)\| = |x - y|$. Show that f is surjective. *Hint*: Argue by contradiction and assume that there is $x_0 \in K \setminus f(K)$. Set $\alpha = d(x_0, f(K))$. Then $\alpha > 0$ (why?). Define the sequence $(x_k) \subset f(K)$ by $x_{k+1} = f(x_k)$ for $k \geq 0$. Show that $\|x_k - x_m\| \geq \alpha$ for all $k, m \in \mathbb{N}$. Reach a contradiction.