Problem 1.

(a) Prove that the union of any (even infinite) number of open sets is open.

(b) Prove that the intersection of two (and hence finitely many) open sets is open.

(c) Give an example of infinitely many open sets whose intersection is not open.

Problem 2. If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, define the distance of $x$ to $A$ by

$$d(x, A) := \inf_{y \in A} |x - y|.$$  

(a) Prove that $|d(x, A) - d(y, A)| \leq |x - y|.$

(b) Prove that $\overline{A} = \{x \in \mathbb{R}^n | d(x, A) = 0\}$.

Problem 3. Let $V = (-1, 0) \cup (0, 1)$. Show that the sets $(-1, 0)$ and $(0, 1)$ are open and closed in $V$. Show that the set $A = (-1/2, 1/2) \cap V$ is open but not closed in $V$.

Problem 4. Let $S \subset \mathbb{R}^n$ and let $f = (f_1, \ldots, f_m) : S \to \mathbb{R}^m$. Prove that $f$ is continuous on $S$ if and only if every $f_j : S \to \mathbb{R}$ is continuous on $S$.

Problem 5.

(a) Prove that if $A \subset \mathbb{R}^n$ is closed and $x \notin A$, then there is $r > 0$ such that $|x - y| \geq r$ for all $y \in A$.

(b) Prove that if $A \subset \mathbb{R}^n$ is closed and $B \subset \mathbb{R}^n$ is compact, and $A \cap B = \emptyset$, then there is $r > 0$ such that $|x - y| \geq r$ for all $x \in A$ and all $y \in B$.

(c) Give an example in $\mathbb{R}^2$ of two closed sets which are not compact so that (b) does not hold.