5 Sequences

Let \((X, d)\) be a metric space. A \textit{sequence} in \(X\) is a function from \(\mathbb{N}\) to \(X\). If \(\varphi : \mathbb{N} \to X\) is a sequence we write

\[(x_n), \ (x_n)_{n \in \mathbb{N}} \text{ or } (x_0, x_1, \ldots)\]

for \(\varphi\), where \(x_n = \varphi(n)\) is the \(n\)th term of the sequence \(\varphi = (x_0, x_1, \ldots)\). Sequences in \(\mathbb{R}\) are called \textit{real sequences}.

**Definition 5.1.** A sequence \((x_n)\) in \(X\) \textit{converges} to \(x\) if for every neighborhood \(U\) of \(x\) there exists \(N \in \mathbb{N}\) such that \(x_n \in U\) for all \(n \geq N\).

In this case we write

\[x_n \to x \quad \text{or} \quad \lim_{n \to \infty} x_n = x \quad \text{or} \quad \lim x_n = x.\]

A sequence \((x_n)\) which does not converge is called \textit{divergent}.

**Proposition 5.2.** The following are equivalent.

(i) \(\lim x_n = x\).

(ii) For every \(\varepsilon > 0\), there is \(N \in \mathbb{N}\) such that \(x_n \in B(x, \varepsilon)\) for all \(n \geq N\).

(iii) For every \(\varepsilon > 0\), there is \(N \in \mathbb{N}\) such that \(d(x_n, x) < \varepsilon\) for all \(n \geq N\).

**Proposition 5.3.** (i) \textit{(Uniqueness of the limit)} A sequence cannot have more than one limit.

(ii) \textit{(Boundedness)} If \((x_n)\) converges to \(x\), then there is \(r > 0\) such that \(x_n \in B_r(x)\).

**Proof of (ii).** Assume that \((x_n)\) converges to \(x\). By (iii) of the previous proposition with \(\varepsilon = 1\), there is \(N\) so that \(x_n \in B_1(x)\) for all \(n \geq N\). Now take \(r\) be any number bigger or equal to \(d(x_1, x), \ldots, d(x_{N-1}, x)\) and 1. Then \(x_n \in B_r(x)\) for all \(n\), as claimed. \(\blacksquare\)

**Definition 5.4.** Let \(\varphi = (x_n)\) be a sequence in \(X\) and let \(\phi : \mathbb{N} \to \mathbb{N}\) be a strictly increasing function. Then \(\varphi \circ \phi : \mathbb{N} \to X\) is called a \textit{subsequence} of \(\varphi = (x_n)\).

We write \((x_n_k)\) for the subsequence \(\varphi \circ \phi\) where \(\phi(k) = n_k\). Since \(\psi\) is strictly increasing \(n_1 < n_2 < n_3 < \ldots\)
Proposition 5.5. If \((x_n)\) is a convergent sequence with limit \(x\), then every subsequence \((x_{n_k})\) of \((x_n)\) converges to \(x\).

Proof. Let \((x_{n_k})\) be a subsequence of \((x_n)\). Take a neighborhood \(U\) of \(x\). Since \(\lim x_n = x\), there is \(N \in \mathbb{N}\) such that \(x_n \in U\) for all \(n \geq N\). From the definition of a subsequence, \(n_k \geq k\) for all \(k \in \mathbb{N}\). So, \(n_k \geq N\) for all \(k \geq N\). Thus \(x_{n_k} \in U\) for all \(k \geq N\). Hence \((x_{n_k})\) converges to \(x\) as claimed. ■

Definition 5.6. A sequence \((x_n)\) in a metric space \((X,d)\) is said to be Cauchy if for every \(\varepsilon > 0\), there exists \(N\) such that \(d(x_n,x_m) \leq \varepsilon\) for all \(n,m \geq N\).

Proposition 5.7. Let \((X,d)\) be a metric space.

(i) Every Cauchy sequence is bounded.

(ii) Every convergent sequence in \(X\) is Cauchy.

Proof of (ii). Assume that \((x_n)\) converges to \(x\). Then, given \(\varepsilon > 0\), there is \(N\) such that \(d(x_n,x) < \varepsilon/2\) for all \(n \geq N\).

Hence \(d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon\) for all \(n,m \geq N\). So, \((x_n)\) is a Cauchy sequence as claimed. ■

5.1 Real sequences

Proposition 5.8. Let \((x_n)\) and \((y_n)\) be convergent sequences in \(\mathbb{R}\) and let \(\alpha \in \mathbb{R}\). Then

(i) The sequence \((x_n + y_n)\) converges and \(\lim(x_n + y_n) = \lim x_n + \lim y_n\).

(ii) The sequence \((\alpha x_n)\) converges and \(\lim(\alpha x_n) = \alpha \lim x_n\).

Proof of (i). Take \(\varepsilon > 0\). Since \((x_n)\) and \((y_n)\) converge to \(x\) and \(y\), respectively, there are \(N_1\) and \(N_2\) such that

\[|x_n - x| < \varepsilon/2 \text{ for } n \geq N_1 \text{ and } |y_n - y| < \varepsilon/2 \text{ for } n \geq N_2.\]

Consequently,

\[|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon\]

for all \(n \geq N = \max\{N_1, N_2\}\). Hence the sequence \((x_n + y_n)\) converges and \(\lim(x_n + y_n) = \lim x_n + \lim y_n\), as claimed. ■
Note that above proposition holds true if instead of real sequences one considers sequences in a normed space \((X, \|\cdot\|)\). We call a real sequence \((x_n)\) a **null sequence** if it converges to 0. That is, for every \(\varepsilon > 0\), there is \(N \in \mathbb{N}\) such that \(|x_n| < \varepsilon\) for all \(n \geq N\).

**Proposition 5.9.** Let \((x_n)\) and \((y_n)\) be sequences in \(\mathbb{R}\).

(i) If \((x_n)\) is a null sequence and \((y_n)\) is a bounded sequence, then the sequence \((x_ny_n)\) is a null sequence.

(ii) If \((x_n)\) and \((y_n)\) are convergent sequences, then the sequence \((x_ny_n)\) converges and \(\lim(x_ny_n) = \lim x_n \lim y_n\).

(iii) If \((x_n)\) converges to \(x\) and \(x \neq 0\), then almost all terms of \((x_n)\) are nonzero and the sequence \((1/x_n)\) converges to \(1/x\).

**Proof of (i).** Suppose that \(\lim x_n = x\) and \(\lim y_n = y\). By (ii) of Proposition 5.3, \(|x_n| \leq A\) and \(|y_n| \leq B\) for all \(n\). Moreover, given \(\varepsilon > 0\), there are \(N_1\) and \(N_2\) so that

\[|x_n - x| < \varepsilon/2(A+|y|)\] for \(n \geq N_1\) and \(|y_n - y| < \varepsilon/2(A+|y|)\) for \(n \geq N_2\).

Consequently,

\[|(x_ny_n) - (xy)| = |x_n(y_n - y) + y(x_n - x)| \leq |x_n||y_n - y| + |y||x_n - x| \leq A|y_n - y| + |y||x_n - x| \leq A\frac{\varepsilon}{2(A+|y|)} + |y|\frac{\varepsilon}{2(A+|y|)} = \varepsilon\]

for all \(n \geq N = \max\{N_1, N_2\}\). Hence the sequence \((x_ny_n)\) converges and \(\lim(x_ny_n) = \lim x_n \lim y_n\), as claimed. \(\blacksquare\)

**Proposition 5.10.** (i) Let \((x_n)\) and \((y_n)\) be convergent sequences in \(\mathbb{R}\) and \(x_n \leq y_n\) for infinitely many \(n\). Then \(\lim x_n \leq \lim y_n\).

(ii) Let \((x_n)\) and \((y_n)\) and \((z_n)\) be sequences in \(\mathbb{R}\) and \(x_n \leq y_n \leq z_n\) for almost all \(n\). If \(\lim x_n = \lim z_n = a\), then \((y_n)\) converges to \(a\).

**Proof of (ii).** Take \(\varepsilon > 0\). Then here are \(N_1\) and \(N_2\) so that

\[|x_n - a| < \varepsilon\] for \(n \geq N_1\) and \[|z_n - a| < \varepsilon\] for \(n \geq N_2\).

Hence

\[a - \varepsilon < x_n < a + \varepsilon\] for \(n \geq N_1\) and \[a - \varepsilon < z_n < a + \varepsilon\] for \(n \geq N_2\)
and since \( x_n \leq y_n \leq z_n \),

\[
a - \varepsilon < x_n \leq y_n \leq z_n < a + \varepsilon
\]

for all \( n \geq N = \max\{N_1, N_2\} \). This means that \( |y_n - a| < \varepsilon \) for all \( n \geq N \). \( \blacksquare \)

**Definition 5.11.** A sequence \( (x_n) \) in \( \mathbb{R} \) is increasing, if \( x_n \leq x_{n+1} \) for all \( n \in \mathbb{N} \), and decreasing if \( x_{n+1} \leq x_n \) for all \( n \in \mathbb{N} \). If \( (x_n) \) is either increasing or decreasing, then it is called monotone.

**Proposition 5.12.** Every increasing (or decreasing) bounded sequence \( (x_n) \) in \( \mathbb{R} \) converges, and

\[
x_n \rightarrow \sup\{x_n \mid n \in \mathbb{N}\} \quad \text{or} \quad x_n \rightarrow \inf\{x_n \mid n \in \mathbb{N}\}.
\]

**Proof.** Suppose that \( (x_n) \) is increasing and Set \( a = \sup\{x_n \mid n \in \mathbb{N}\} \in \mathbb{R} \). By the characterization of the supremum,

(a) \( x_n \leq a \) for all \( n \).

(b) given \( \varepsilon \), there is \( N \) such that

\[
a - \varepsilon < x_N \leq x_n \leq a + \varepsilon
\]

for all \( n \geq N \). Hence \( (x_n) \) converges to \( x\) as claimed. \( \blacksquare \)

**Example 5.13.** Let \( a > 0 \) and \( \alpha > 0 \). Define a sequence \( (x_n) \) by setting \( x_0 = \alpha \) and

\[
x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right). \tag{1}
\]

We claim that the sequence \( (x_n) \) converges to \( \sqrt{a} \). To see this we shall show that \( (x_n) \) is bounded below by \( \sqrt{a} \) and it is decreasing. Consider the function \( f(x) = \frac{1}{2} (x + \frac{a}{x}) \) for \( x > 0 \). Differentiating one finds that \( f'(x) = \frac{1}{2} \left( 1 - \frac{a^2}{x^2} \right) < 0 \) for \( x \in (0, \sqrt{a}) \) and \( f'(x) > 0 \) for \( x \in (\sqrt{a}, \infty) \). Hence \( f \) takes its minimum at \( a = \sqrt{a} \) and \( f(x) \geq f(\sqrt{a}) = \sqrt{a} \). So, \( x_n \geq \sqrt{a} \) for all \( n \geq 1 \). Next

\[
x_n - x_{n+1} = x_n - \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) = \frac{1}{2} \left( x_n - \frac{a}{x_n} \right) = \frac{x_n^2 - a}{2x_n} \geq 0
\]

since \( x_n^2 \geq a \). This means that \( x_n \geq x_{n+1} \) for all \( n \geq 1 \) so that \( (x_n) \) is decreasing. Then the above proposition implies that \( x_n \rightarrow x \) where \( x \geq \sqrt{a} > 0 \) Then also \( x_{n+1} \rightarrow x \). So, taking limits in (1), we get

\[
x = \frac{1}{2} \left( x + \frac{a}{x} \right)
\]

and solving for \( x \) one finds \( x = \sqrt{a} \).
Proposition 5.14. Every sequence contains a monotone subsequence.

Proof. Consider a sequence \((x_n)\). We call the \(m\)-th term \(x_m\) a peak if \(x_m \geq x_n\) for all \(n \geq m\). There are two cases to consider.

Case 1. There are infinitely many peaks. List them according the increasing subscripts: \(x_{n_1}, x_{n_2}, \ldots\). Then, since each of them is a peak, we have
\[
x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \ldots
\]
Hence \((x_{n_k})\) is a decreasing subsequence of \((x_n)\).

Case 2. There are finitely many peaks. List them according to the increasing \(x_{m_1}, x_{m_2}, \ldots, x_{m_l}\). Then set \(n_1 = m_l + 1\). The \(n_1\) terms \(x_{n_1}\) is not a peak and so there is an index \(n_2 > n_1\) such that \(x_{n_1} < x_{n_2}\). The term \(x_{n_2}\) is also not a peak and there is \(n_3 > n_2\) such that \(x_{n_2} < x_{n_3}\). Continuing this way we obtain a strictly increasing subsequence \((x_{n_k})\) of \((x_n)\). ■

5.1.1 Infinite limits

Definition 5.15. A sequence of real numbers \((x_n)\) converges to \(\infty\), in symbols \(x_n \to \infty\) or \(\lim x_n = \infty\), if for every \(M > 0\) there exists \(N\) such that
\[
x_n \geq M \quad \text{for all } n \geq N.
\]
Similarly, \((x_n)\) converges to \(-\infty\), in symbols \(x_n \to -\infty\) or \(\lim x_n = -\infty\) if for every \(M < 0\), there is there exists \(N\) such that
\[
x_n \leq M \quad \text{for all } n \geq N.
\]

5.1.2 Some Special Sequences

(1) Let \(a \in \mathbb{R}\). Then
\[
a^n \to 0 \quad \text{if } |a| < 1
\]
\[
a^n \text{ diverges} \quad \text{if } |a| > 1
\]
Suppose that \(|a| < 1\). Then \(|a| = \frac{1}{1+h}\) for some \(h > 0\). By the binomial theorem,
\[
(1+h)^n = \sum_{k=0}^{n} \binom{n}{k} h^k \geq nh
\]
so that
\[
0 \leq |a^n| = |a|^n = \frac{1}{(1+h)^n} \leq \frac{1}{nh}.
\]
Since $1/nh \rightarrow 0$, $a^n \rightarrow 0$ by the comparison test. Assume that $|a| > 1$. Then $|a| = 1 + h$ for some $h$ and then

$$|a^n| = |a^n| = (1 + h)^n \geq nh,$$

showing that $(a^n)$ is unbounded.

(2) Let $k \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. Then

$$\lim_{n \rightarrow \infty} n^k a^n = 0.$$

To see this write $a = 1 + h$ where $h > 0$ and apply the binomial theorem,

$$a^n = (1 + h)^n = \sum_{m=0}^{n} \binom{n}{m} h^m \geq \binom{n}{k+1} h^{k+1} \geq \frac{n(n-1) \cdots (n-k)}{(k+1)!} h^{k+1}.$$

So,

$$0 \leq \frac{n^k}{a^n} \leq \frac{(k+1)! n^k}{h^{k+1} n(n-1) \cdots (n-k)},$$

and the claimed follows using the comparison since the right hand side converges to 0 as $n \rightarrow \infty$.

(3) For all $a \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{a^n}{n^n} = 0.$$

(4) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

To see this write $\sqrt[n]{n} = 1 + a_n$ with $a_n \geq 0$ (since it is easy to see that $\sqrt[n]{n} \geq 1$. So,

$$(1 + a_n)^n = n.$$

By the binomial theorem,

$$n = (1 + a_n)^n = 1 + \binom{n}{1} a_n + \binom{n}{2} a_n^2 + \cdots \geq 1 + \binom{n}{2} a_n^2 = 1 + \frac{n(n-1)}{2} a_n^2.$$

From this we get

$$0 \leq a_n^2 \leq \frac{2}{n}, \quad \text{i.e.,} \quad 0 \leq \sqrt[n]{n} - 1 = a_n \leq \sqrt{2/n}$$

which by comparison implies that $\sqrt[n]{n} \rightarrow 1$.

(5) For all $a > 0$, $\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$.

If $a = 1$, then there is nothing to prove. If $a > 1$, then

$$1 \sqrt[n]{a} \leq \sqrt[n]{a}$$

which implies that $(\sqrt[n]{a})$ converges to 1. So, assume that $0 < a < 1$, then $1/a > 1$ so that $\sqrt[1/a]{a} \rightarrow 1$. Since $\sqrt[n]{a} = 1/\sqrt[1/a]{a}$, the claim follows.
5.1.3 The limit superior and limit inferior

Let \((x_n)\) be a bounded sequence. For each \(n \in \mathbb{N}\), let

\[ y_n = \sup_{m \geq n} x_m = \sup \{x_n, x_{n+1}, \ldots\}. \]

Then \(\{y_n\}\) is decreasing, and it is bounded since \((x_n)\) is bounded. By Proposition 5.12, \((y_n)\) converges. The limit is called the upper limit of \((x_n)\), and is denoted by \(\lim sup x_n\). Similarly, let \(z_n = \inf_{m \geq n} x_m = \inf \{x_n, x_{n+1}, \ldots\}\). Then \((z_n)\) is increasing, and it is bounded since \((x_n)\) is bounded. The limit of \((z_n)\) is called the lower limit of \((x_n)\), and is denoted by \(\lim inf x_n\).

If \((x_n)\) is not bounded from above, then its upper limit is equal to \(\infty\) and if \((x_n)\) is not bounded from below, then its lower limit is equal to \(-\infty\).

The basic properties of the upper and the lower limits are listed in the following proposition:

**Proposition 5.16.** If \((x_n)\) and \((y_n)\) are sequences of real numbers, then:

(a) \(\lim sup(-x_n) = -\lim inf x_n\) and \(\lim inf(-x_n) = -\lim sup x_n\).

(c) \(\lim sup(ax_n) = a\lim sup x_n\) and \(\lim inf(ax_n) = a\lim inf x_n\) for any \(a > 0\).

(d) \(\lim sup(x_n + y_n) \leq \lim sup x_n + \lim sup y_n\) and \(\lim inf x_n + \lim inf y_n \leq \lim inf(x_n + y_n)\).

(e) \(\lim inf x_n \leq \lim sup x_n\), with equality if and only if \((x_n)\) converges. In this case \(\lim sup x_n = \lim inf x_n\).

**Theorem 5.17** (Bolzano-Weierstrass Theorem).
Let \((x_n)\) be a bounded sequence in \(\mathbb{R}\). Then \((x_n)\) has a convergent subsequence.

**Proof.** By Proposition 5.14, \((x_n)\) has a monotone subsequence \((x_{n_k})\). Since \((x_{n_k})\) is bounded, Proposition 5.12 implies that it converges. \(\square\)

**Theorem 5.18** (Cauchy’s criterion). A sequence \((x_n)\) of real numbers converges if and only if \((x_n)\) is a Cauchy sequence.

**Proof.** We already know that a convergent sequence is Cauchy (Proposition 5.7). Hence we have to prove that a Cauchy sequence \((x_n)\) converges to some \(x \in \mathbb{R}\). By Proposition 5.7, \((x_n)\) is bounded so that by Theorem 5.17, it contains a convergent subsequence, say \(x_{n_k} \to x\). We claim that \(x_n \to x\). To see this, take \(\varepsilon > 0\). Then there is \(K\) such that

\[ |x_{n_k} - x| < \varepsilon / 2 \quad \text{for all} \quad k \geq K. \]
Moreover, since \((x_n)\) is Cauchy, there is \(N\) such that

\[|x_n - x_m| < \varepsilon/2\quad \text{for all } n, m \geq N.\]

Take \(k\) such that \(n_k \geq N\) and let \(n \geq N\). Then

\[|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.\]

Hence \((x_n)\) converges to \(x\), as claimed. ■

**Example 5.19.** Define \((x_n)\) by setting \(x_1 = 1, x_2 = 2,\) and

\[x_n = \frac{1}{2}(x_{n-1} + x_{n-2}).\]

We claim that the sequence converges to \(x = 5/3\). To see this we shall show that \((x_n)\) is Cauchy and that the subsequence \((x_{2n+1})\) converges to \(x = 5/3\). First note that \(1 \leq x_n \leq 2\) for all \(n\). Next

\[|x_n - x_{n+1}| = \frac{1}{2^n}.\]

Indeed, for \(n = 1, |x_1 - x_2| = 1 = \frac{1}{2^n}\). Assuming that \(|x_n - x_{n+1}| = \frac{1}{2^n}\), we show that \(|x_{n+1} - x_{n+2}| = \frac{1}{2^n}\). We have

\[|x_{n+1} - x_{n+2}| = |x_{n+1} - \frac{1}{2}(x_{n+1} - x_n)| = \frac{1}{2}|x_n - x_{n+1}| = \frac{1}{2^n},\]

as claimed. Consequently, if \(m > n,\)

\[|x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \ldots + |x_{m-1} - x_m|\]

\[\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{m-1}}\]

\[= \frac{1}{2^n} \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-n-1}}\right) < \frac{1}{2^{n-2}},\]

and since \(1/2^{n-2} \to 0\), it follows that indeed \((x_n)\) is Cauchy. Now to find the limit of \(x\) we first show that \(x_{2n-1} < x_{2n}\). This holds true for \(n = 1\). Assuming that \(x_{2n-1} < x_{2n}\), we show that \(x_{2n+1} < x_{2n+2}\). From the assumption if follows that

\[x_{2n+1} < \frac{1}{2}(x_{2n} + x_{2n-1}) < x_{2n}.\]

So, \(x_{2n-1} < x_{2n+1}\) and

\[x_{2n+2} = \frac{1}{2}(x_{2n+1} + x_{2n}) > \frac{1}{2}(x_{2n-1} + x_{2n}) = x_{2n+1},\]

as claimed. From this and \(|x_{2n+1} - x_{n+2}| = 1/2^{2n}\), it follows that

\[x_{2n+2} = x_{2n+1} + \frac{1}{2^{2n}}.\]
Now use induction to prove that

\[ x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{2n-1}}. \]

For \( n = 1 \) this is clear. Assuming the above equality we show that \( x_{2n+3} = 1 + \frac{1}{2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{2n+1}} \). We have

\[
x_{2n+3} = \frac{1}{2}(x_{2n+2} + x_{2n+1}) = \frac{1}{2} \left( \frac{1}{2^{2n}} + 2x_{2n+1} \right)
\]

\[
= 1 + \frac{1}{2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{2n-1}} + \frac{1}{2^{2n+1}},
\]

as claimed. Now, note that

\[
x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{2n-1}}
\]

\[
= 1 + \frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{2n-2}} \right)
\]

\[
= 1 + \frac{1}{2} \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^{n-1}} \right)
\]

\[
= 1 + \frac{2}{3} \left( 1 - \frac{1}{4^n} \right) \rightarrow \frac{5}{3},
\]

where we have used the formula \( \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \) provided that \( a \neq 1 \).