Math 250
Fall 2007
Exam 2

NAME: ____________________

ID No: ____________________

SECTION: ____________________

This exam contains 10 questions on 10 pages (including this title page). This exam is worth a total of 100 points. The exam is broken into two parts. There are six multiple choice questions, each worth 5 points, and 4 partial credit problems. To receive full credit for a partial credit problem all work must be shown. When in doubt, fill in the details.

No notes, books or calculators may be used during the exam.

Please, Box Your Final Answer (when possible).

1:	C
2:	D
3:	A
4:	D
5:	D
6:	D
7:	
8:	
9:	
10:	
Total:	

Multiple Choice Section

1. *(5 points)* Suppose the Wronskian of two functions f and g is

$$W(f, g) = \sin(x).$$

Which of the follwing statements is FALSE?

(a) f and g are linearly independent on any open interval.

(b) f and g are linearly independent on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

✓ (c) f and g can be solutions to a second order linear homogeneous differential equation on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(d) f and g can be solutions to a second order linear homogeneous differential equation on the interval $(0, \pi)$.

2. *(5 points)* Which of the following is a suitable form for a particular solution $y(t)$ to the differential equation

$$y'' + 6y' + 9y = 2te^{-3t} + 4e^{-3t}\cos(t) + 5e^{-3t}\sin(t)\ ?$$

A, B, C, D below are constants.

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$r_1 = r_2 = -3$

$y_1 = e^{-3t}, \quad y_2 = te^{-3t}$

(a) $Ate^{-3t} + Be^{-3t}\cos(t) + Ce^{-3t}\sin(t)$

(b) $At^2e^{-3t} + Be^{-3t}\cos(t) + Ce^{-3t}\sin(t)$

(c) $(At^2 + Bt)e^{-3t} + Cte^{-3t}\cos(t) + Dte^{-3t}\sin(t)$

✓ (d) $(At^3 + Bt^2)e^{-3t} + Ce^{-3t}\cos(t) + De^{-3t}\sin(t)$

3. *(5 points)* A spring is stretched L meters by mass of 4 kilograms. The system is set in motion at time $t = 0$ by an external force $F(t) = \sin(\omega t)$ Newtons. Assume no damping and take $g = 10\text{m/sec}^2$. Then resonance will occur when

$4 \cdot 10 = kL$. $k = \frac{40}{L}$

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{40}{L \cdot 4}} = \sqrt{\frac{10}{L}}$

$\omega = \omega_0 = \sqrt{\frac{10}{L}}$

✓ (a) $\omega^2 L = 10$.

(b) $\omega^2 L = 20$.

(c) $\omega^2 L = 30$.

(d) $\omega^2 L = 40$.

4. *(5 points)* If

$$Y_1 = t, \qquad Y_2 = t + 2e^t, \qquad Y_3 = t + e^t + e^{-t}$$

are solutions to a nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t), \quad \text{where } g(t) \neq 0,$$

then the general solution to this differential equation is

(a) $c_1 t + c_2(t + 2e^t) + c_3(t + e^t + e^{-t})$, where c_1, c_2, c_3 are any constants.

(b) $c_1 t + c_2 e^t + c_3 e^{-t}$, where c_1, c_2, c_3 are any constants.

(c) $c_1 t + c_2(t + 2e^t) + t + e^t + e^{-t}$, where c_1, c_2 are any constants.

✓ (d) $c_1 e^t + c_2 e^{-t} + t$, where c_1, c_2 are any constants.

5. *(5 points)* A spring-mass system has mass 4kg and spring constant 9m/sec^2. What is the critical value of the damping constant γ (the value for which the system goes from underdamped to overdamped state)?

(a) $\gamma = 4$ kg/sec

(b) $\gamma = 6$ kg/sec

(c) $\gamma = 9$ kg/sec

✓ (d) $\gamma = 12$ kg/sec

$m = 4, \quad k = 9.$

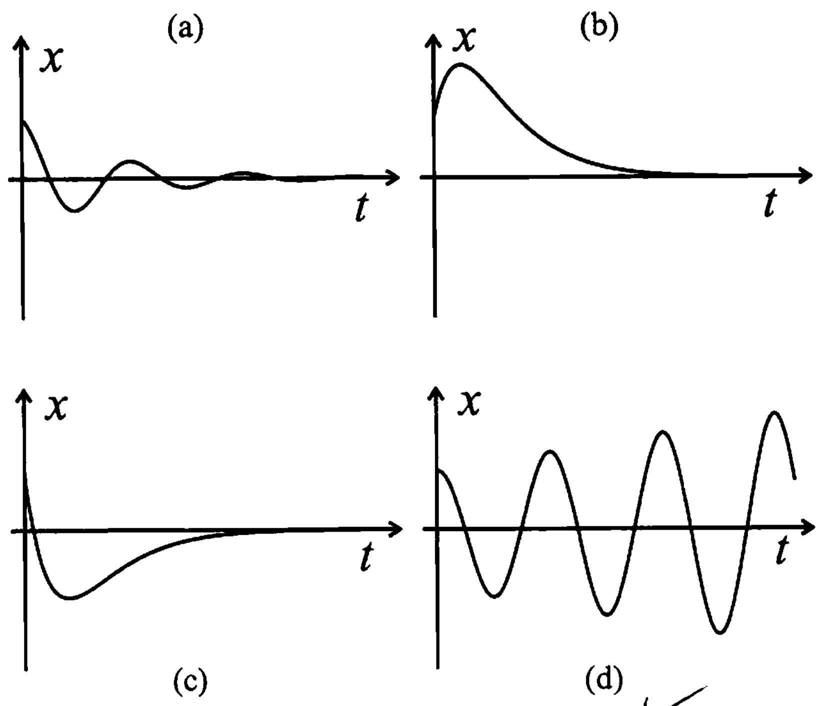
$4r^2 + \gamma r + 9 = 0$. has repeated roots

$\Leftrightarrow \gamma^2 = 4 \cdot 4 \cdot 9$

$\Leftrightarrow \gamma = 4 \cdot 3 = 12$

6. *(5 points)* Which of the following CANNOT be the graph of a solution to a differential equation of the form

$$mx'' + \gamma x' + kx = 0, \quad m, \gamma, k > 0 \ ?$$



✓

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = C e^{-\int p(x)\,dx}$$

$$\shortparallel$$

$$y_1 y_2' - y_2 y_1'$$

$$v' y_1^2 = C e^{-\int p(x)\,dx}$$

Partial Credit Section

7. *(15 points)* Given that $y_1 = x^2$ is a solution to

$$x^2y'' - 2xy' + 2y = 0, \quad x > 0,$$

(a) Find the second solution y_2 linearly independent of y_1. Explain why y_1 and y_2 are linearly independent.

1 pt

Write $y_2 = v y_1$ $\qquad y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0, \quad x > 0$

$W(y_1, y_2) = W(y_1, vy_1) = v' y_1^2 = C e^{-\int p(x)\,dx} = C \cdot e^{\int \frac{2}{x} dx}$

$v' x^4 = C e^{2 \ln x} = C x^2.$

9 pts $\qquad v' = \frac{C}{x^2}.$

2 pts $\qquad v = \int \frac{C}{x^2} dx = C \int x^{-2} dx = C \cdot \frac{x^{-1}}{-1} + c' = -Cx^{-1} + c'.$

1 pt $\qquad$ Choose $C = -1$, $c' = 0$ to get $v = x^{-1}$

1 pt $\qquad y_2 = x^{-1} \cdot x^2 = x.$

1 pt $\qquad y_1$ and y_2 are linearly indep. since $W(y_1, y_2) \neq 0$.

(b) Find the general solution to this differential equation.

1 pt

Gen'l sol'n $y = c_1 x^2 + c_2 x$, $\quad c_1, c_2$ consts.

8. *(20 points)* Consider the nonhomogeneous differential equation

$$y'' - 4y' + 5y = te^{2t} + 3.$$

(a) Find two linearly independent solutions of the corresponding homogeneous differential equation.

4 pts

Char eq'n $r^2 - 4r + 5 = 0$. $r_1, r_2 = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

$\lambda = 2, \quad \mu = 1$

$y_1 = e^{2t} \cos t. \quad y_2 = e^{2t} \sin t.$

y_1, y_2 lin. indep. since they are not proportional; or $W(y_1, y_2) \neq 0$

(b) Find a particular solution of the nonhomogeneous equation.

16

3 pts Try $Y(t) = (At + B)e^{2t} + C$.

1 $Y' = Ae^{2t} + 2(At+B)e^{2t} = (2At + A + 2B)e^{2t}$.

1 $Y'' = 2Ae^{2t} + 2(2At + A + 2B)e^{2t} = e^{2t}(4At + 4A + 4B)$

Y is a sol'n

$\Leftrightarrow e^{2t}(4At + 4A + 4B) - 4e^{2t}(2At + A + 2B) + 5(At+B)e^{2t} + 5C$

$= te^{2t} + 3$.

i.e. $e^{2t}(At + B) + 5C = te^{2t} + 3$.

$At + B = t$

$5C = 3$

$\therefore A = 1, \quad B = 0, \quad C = \frac{3}{5}$.

2 2 1

7 pts Solve for A, B, C.

1 pt $Y(t) = te^{2t} + \frac{3}{5}$.

If set-up misses B, lose 3 pts.

1

(c) Find the general solution of the nonhomogeneous equation.

$$y = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t + t e^{2t} + \frac{3}{5}, \quad c_1, c_2 \text{ consts.}$$

4

(d) Give the *form* of a particular solution to the equation

$$y'' - 4y' + 5y = 3e^{2t}\cos(t).$$

Do NOT solve for it.

$$Y(t) = Ate^{2t}\cos t + Bte^{2t}\sin t, \qquad A, B \text{ consts.}$$

9. *(20 points)* A mass of 2kg stretches a spring 10cm. The mass is pulled down 20cm from the equilibrium position and then released with downward initial veolicity 2m/sec. Ignore air resistance and take $g = 10\text{m/sec}^2$.

5 (a) Write down the *differential equation* governing the motion of the mass.

$m = 2, \quad L = 0.1 . \qquad 2 \cdot 10 = k \cdot 0.1 \qquad k = 200 .$ 1 pt

$u(t)$ = position of the mass from equilibrium after t sec.

2 1. Differential Equation: $2u'' + 200u = 0$

2 2. Initial Conditions: $u(0) = 0.2 , \quad u'(0) = 2$

6 (b) Determine the position of the mass at any time t.

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{2}} = \sqrt{100} = 10.$

$u_1 = \cos 10t , \qquad u_2 = \sin 10t$

$u = C_1 \cos 10t + C_2 \sin 10t$

$u' = -10 C_1 \sin 10t + 10 C_2 \cos 10t .$

$u(0) = C_1 = 0.2 . \qquad u'(0) = 10 C_2 = 2 . \qquad C_2 = 0.2 .$

$u(t) = 0.2 \cos 10t + 0.2 \sin 10t$

$\neq$

6 (c) Find the amplitude, frequency, period, and the phase of the motion.

$u(t) = 0.2 \cos 10t + 0.2 \sin 10t = R \cos(10t - \delta)$, where

$R = \sqrt{(0.2)^2 + (0.2)^2} = 0.2\sqrt{2} \overset{m}{=} \text{amplitude}$

frequency $= 10$ rad/sec.

period $= \frac{2\pi}{10}$

$\left.\begin{array}{l} \cos\delta = \frac{0.2}{R} \\ \sin\delta = \frac{0.2}{R} \end{array}\right\} \Rightarrow \delta$ lies in 1st quadrant

$\tan\delta = 1$. $\quad \delta = \arctan 1 = \frac{\pi}{4} = \text{phase}$

1 1. Amplitude: $0.2\sqrt{2}$

1 2. Frequency: 10

1 3. Period: $\frac{2\pi}{10}$

3 4. Phase: $\frac{\pi}{4}$

(d) Find the first time the mass crosses the equilibrium position.

3 Solve $u(t) = 0$. $\quad 10t - \delta = \frac{\pi}{2}$, $\quad 10t = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

Ans. $t = \frac{3\pi}{40}$ sec.

10. *(15 points)* Find the Laplace transform of the function

$$f(t) = \begin{cases} 5 & \text{if } 0 \le t < 6, \\ t-1 & \text{if } t \ge 6 \end{cases}$$

from definition.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt \qquad 2\text{ pts}$$

$$= \int_0^6 5e^{-st}\, dt + \lim_{A\to\infty} \int_6^A (t-1)\, e^{-st}\, dt \qquad 3\text{ pts}$$

$u = t-1 \quad dv = e^{-st}\,dt$

$du = dt \quad v = \frac{e^{-st}}{-s}$

$$= 5 \cdot \frac{e^{-st}}{-s}\Big|_0^6 \cancel{+ \lim_{A\to\infty}}$$

$$+ \lim_{A\to\infty} (t-1)\frac{e^{-st}}{-s}\Big|_6^A + \int_6^A \frac{e^{-st}}{+s}\, dt$$

3 pts
$$= \underbrace{\frac{5e^{-6s}}{-s} - \frac{5}{-s}} + \lim_{A\to\infty} \underbrace{(A-1)\frac{e^{-sA}}{-s}}_{\downarrow\; 0 \text{ for } s>0} - 5\cdot\frac{e^{-6s}}{-s} + \frac{e^{-st}}{-s^2}\Big|_6^A$$

$$= -\frac{5e^{-6s}}{s} + \frac{5}{s} + \frac{5e^{-6s}}{s} + \lim_{A\to\infty} \underbrace{\frac{e^{-sA}}{-s^2}}_{\downarrow\; 0 \text{ for } s>0} - \frac{e^{-6s}}{-s^2}$$

1 pt.
$$= \frac{5}{s} + \frac{e^{-6s}}{s^2}, \quad s > 0.$$

First $\int$ 3 pts

2nd $\int$ 6 pts

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