MATH 250

Final Exam

December 16, 2005

Name:________________

Student Number:________________

Instructor:________________

Section:________________

This exam has 16 questions for a total of 150 points. There are 6 partial credit questions. **In order to obtain full credit for partial credit problems, all work must be shown. Credit will not be given for an answer not supported by work.**

THE USE OF CALCULATORS or ANY OTHER ELECTRONIC DEVICES IS NOT PERMITTED IN THIS EXAMINATION.
At the end of the examination, the booklet will be collected. The last sheet of the booklet is a table of Laplace transforms and can be removed. **Be careful to remove only the last page of the examination.**

Do not write in this box.

1:	d
2:	a
3:	a
4:	c
5:	d
6:	c
7:	c
8:	c
9:	d
10:	
11:	
12:	
13:	
14:	
15:	
16:	
Total:	

1. (5 points) The autonomous equation $y' = y^2 - y - 2$ has the following equilibrium solutions:

   (a) unstable $y = 2$ and asymptotically stable $y = 1$;

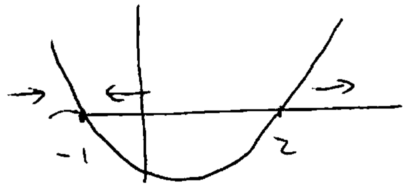
   (b) asymptotically stable $y = 2$ and unstable $y = -1$;

   (c) semistable $y = 2$ and asymptotically stable $y = -1$;

   (d) unstable $y = 2$ and asymptotically stable $y = -1$.

$(y-2)(y+1)$

$y = 2, -1$



2. (5 points) On which of the following intervals is the unique solution of the initial value problem

$$(\sin t)y' - \frac{t}{t-1}y = \frac{1}{t-2}, \quad y(3) = 5$$

certain to exist?

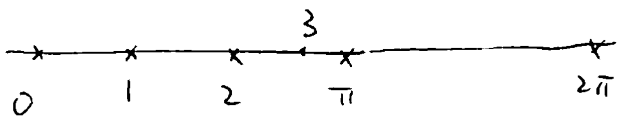
   (a) $(2,\ \pi)$;

   (b) $(1,\ 2)$;

   (c) $(2,\ \infty)$;

   (d) $(0,\ \pi)$.

$y' - \frac{t}{t-1}\frac{1}{\sin t}\, y = \frac{1}{t-2}\frac{1}{\sin t}$



3. (5 points) Which of the following statements best describes the behavior of a nonzero solution to

$$y'' - 2y' + 2y = 0?$$

$r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

✓ (a) unbounded oscillations;

(b) bounded periodic oscillations;

(c) bounded non-periodic oscillations;

(d) $\lim_{t\to\infty} y(t) = 0$.

4. (5 points) Find a suitable form of a particular solution to

$$y'' - 3y' - 4y = 3e^{4t} + t - 8e^{-t}\cos t$$

with constants A, B, C, D, and E.

$r^2 - 3r - 4 = 0 \quad (r-4)(r+1) = 0$.

$r = 4, \ -1$.

$y_1 = e^{+4t}, \quad y_2 = e^{-t}$.

(a) $Ae^{4t} + Bt + Ce^{-t}\cos t$;

(b) $Ae^{4t} + Bt + C + De^{-t}\cos t + Ee^{-t}\sin t$;

✓ (c) $Ate^{4t} + Bt + C + De^{-t}\cos t + Ee^{-t}\sin t$;

(d) $Ate^{4t} + Bt + C + Dte^{-t}\cos t + Ete^{-t}\sin t$;.

$Y = Ate^{4t} + Bt + C + De^{-t}\cos t + Ee^{-t}\sin t$

5. (5 points) What is the Laplace transform of

$$f(t) = u_3(t)e^{2t}?$$

$$= u_3(t)\, e^{2(t-3)+6}$$

(a) $\dfrac{e^{-3s}}{s}\dfrac{1}{s-2}$.

(b) $e^{-3s}\dfrac{1}{s-2}$.

(c) $e^{-3s}\dfrac{e^3}{s-2}$.

✓(d) $e^{-3s}\dfrac{e^6}{s-2}$.

6. (5 points) Consider the function

$$f(t) = t^2 + u_1(t)(2t-1) + u_3(t)(2e^t).$$

What is $f(2)$?

$4 + (4-1)$

(a) 3;

(b) 4;

✓(c) 7;

(d) $7 + 2e^2$.

7. (5 points) Find the system which is equivalent to the second order linear differential equation

$$tu'' + (\sin t)u' - 5u = t^2.$$

$x_1 = u, \quad x_2 = u' = x_1'$

$x_1' = x_2.$

$x_2' = u'' = -\frac{\sin t}{t} u' + \frac{5}{t} u + t$

$= \frac{5}{t} x_1 - \frac{\sin t}{t} x_2 + t.$

(a) $\begin{cases} x_1' = x_2 \\ x_2' = -(\sin t)x_1 + 5x_2 + t^2. \end{cases}$

(b) $\begin{cases} x_1' = x_2 \\ x_2' = 5x_1 - (\sin t)x_2 + t^2. \end{cases}$

✓(c) $\begin{cases} x_1' = x_2 \\ x_2' = \frac{5}{t}x_1 - \frac{\sin t}{t}x_2 + t \end{cases}$

(d) $\begin{cases} x_1' = x_2 \\ x_2' = -\frac{\sin t}{t}x_1 + \frac{5}{t}x_2 + t \end{cases}$

8. (5 points) Let $\boldsymbol{A}$ be a real 2×2 matrix which has an eigenvalue $-1+2i$ and corresponding eigenvector $\begin{pmatrix} 1 \\ 2-3i \end{pmatrix}$. Find the general solution of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with constants c_1 and c_2.

$\vec{z} = e^{(-1+2i)t}\begin{pmatrix} 1 \\ 2-3i \end{pmatrix}$

$= e^{-t}(\cos 2t + i \sin 2t)\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i\begin{pmatrix} 0 \\ -3 \end{pmatrix}\right)$

$= e^{-t}\left(\cos 2t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -3 \end{pmatrix}\right)$

$+ i e^{-t}\left(\cos 2t \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$

(a) $c_1 e^{-t}\begin{pmatrix} \cos 2t \\ 3\cos 2t + 2\sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin 2t \\ 3\sin 2t - 2\cos 2t \end{pmatrix}$;

(b) $c_1 e^{-t}\begin{pmatrix} \cos 2t \\ 2\cos 2t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 0 \\ -3\sin 2t \end{pmatrix}$;

✓(c) $c_1 e^{-t}\begin{pmatrix} \cos 2t \\ 2\cos 2t + 3\sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin 2t \\ 2\sin 2t - 3\cos 2t \end{pmatrix}$;

(d) $c_1 e^{-2t}\begin{pmatrix} \cos t \\ 2\cos t + 3\sin t \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} \sin t \\ 2\sin t - 3\cos t \end{pmatrix}$.

9. (5 points) Let $\boldsymbol{A}$ be a real 2×2 matrix which has a repeated eigenvalue 3. Given that

$$\boldsymbol{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad (\boldsymbol{A} - 3\boldsymbol{I})\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

find the general solution of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with constants c_1 and c_2.

(a) $c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$;

(b) $c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 t e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$;

(c) $c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 t e^{3t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$;

(d) $c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[t e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$.

10. (9 points) Find the solution of the following initial value problem :

$$ty' - 2y = 2t^3e^{2t}, \quad y(1) = \frac{1}{2}e^2.$$

$y' - \frac{2}{t}y = 2t^2e^{2t}$

integrating factor $= e^{\int -\frac{2}{t}\,dt} = e^{-2\ln t} = t^{-2}$ 3

$(t^{-2}y)' = 2e^{2t}$.

$t^{-2}y = \int 2e^{2t}\,dt = e^{2t} + c$

$y = t^2e^{2t} + ct^2$. 4

$\frac{1}{2}e^2 = y(1) = e^2 + c \qquad c = -\frac{1}{2}e^2$.

2

$y = t^2e^{2t} - \frac{1}{2}e^2t^2$.

11. (15 points) Solve the initial value problem

$$y'' + 2y' + y = 1 + 4e^{-t}, \quad y(0) = 2, \quad y'(0) = -1.$$

$r^2 + 2r + 1 = 0$, $(r+1)^2 = 0$, $r = -1, -1$.

(I) $y_1 = e^{-t}$, $y_2 = te^{-t}$. 3 homog.

(II) Try $Y(t) = A + Bt^2e^{-t}$. 3

$Y' = 2Bte^{-t} - Bt^2e^{-t} = (-Bt^2 + 2Bt)e^{-t}$

$Y'' = (-2Bt + 2B)e^{-t} - (-Bt^2 + 2Bt)e^{-t}$

$= (Bt^2 - 4Bt + 2B)e^{-t}$

$Y'' + 2Y' + Y = (Bt^2 - 4Bt + 2B)e^{-t} + 2(-Bt^2 + 2Bt)e^{-t} + A + Bt^2e^{-t}$

$= 2Be^{-t} + A = 1 + 4e^{-t}$ 1. 4.

$\therefore A = 1$, $2B = 4$, $B = 2$

$Y(t) = 1 + 2t^2e^{-t}$

(III) Gen'l soln $y(t) = c_1e^{-t} + c_2te^{-t} + 1 + 2t^2e^{-t}$

$y'(t) = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t} + 4te^{-t} - 2t^2e^{-t}$

$y(0) = c_1 + c_2 + 1 = 2$.

$y'(0) = -c_1 + c_2 = -1$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = -1 \end{cases}$$ 2

$2c_2 = 0$, $c_2 = 0$, $c_1 = 1$

Ans. $y(t) = e^{-t} + 1 + 2t^2e^{-t}$ 2

3 setting up
9 partial ~~credit~~ fractions
3 inversion

12. (10 points) Find the Laplace transform of

$$f(t) = e^t + u_\pi(t)\cos t + u_2(t)(t^2 - 2).$$

$$\cos t = \cos((t-\pi)+\pi) = \cos(t-\pi)\cos\pi - \sin(t-\pi)\sin\pi$$
$$= -\cos(t-\pi)$$

$$t^2 - 2 = (t-2)^2 + 4t - 6 = (t-2)^2 + 4(t-2) + 2$$

$$\cancel{t^2 - 4t + 4}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^t - u_\pi(t)\cos(t-\pi) + u_2(t)((t-2)^2 + 4(t-2) + 2)\}$$

$$= \frac{1}{s-1} - e^{-\pi s}\frac{s}{s^2+1} + e^{-2s}\left(\frac{2}{s^3} + 4\cdot\frac{1}{s^2} + \frac{2}{s}\right)$$

1 3 6

13. (15 points) Find the inverse Laplace transform of

$$F(s) = \frac{5s^3 - 4s^2 + 3s - 1}{(s^2 - s)(s^2 + 1)}.$$

$$\frac{5s^3 - 4s^2 + 3s - 1}{(s^2 - s)(s^2+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}. \qquad 2$$

$$5s^3 - 4s^2 + 3s - 1 = A(s-1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s-1)$$

Set $s = 0$: $\quad -1 = -A, \quad A = 1.$

$s = 1$: $\quad 3 = B \cdot 2 \quad B = \frac{3}{2}.$

~~Compare coeffs of s^3.~~

$s = i$: $\quad -5i + 4 + 3i - 1 = 3 - 2i = (Ci + D)(-1-i)$ $\qquad 9$

$$= (C - D) + i(-C - D)$$

$$\begin{cases} C - D = 3 \\ C + D = +2 \end{cases} \Rightarrow C = \frac{5}{2}, \quad D = -\frac{1}{2}.$$

(Compare coeffs of s^3: $\quad 5 = A + B + C, \quad C = 5 - 1 - \frac{3}{2} = \frac{5}{2}.$

s^2: $\quad -4 = -A - C + D \quad D = -4 + A + C = -4 + 1 + \frac{5}{2} = -\frac{1}{2}$)

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} + \frac{\frac{3}{2}}{s-1} + \frac{\frac{5}{2}s - \frac{1}{2}}{s^2+1} \right\} = 1 + \frac{3}{2}e^t + \frac{5}{2}\cos t - \frac{1}{2}\sin t. \qquad 4$$

14. (20 points) A system with a mass of $1kg$ at the end of a spring with spring constant $13N/m$ is immersed in a medium with damping constant $6Ns/m$. The mass starts at the equilibrium position with an upward velocity $1m/s$.

(a) (5 points) Suppose that an external force of $3N$ is applied to the system from $t = 2s$ to $t = 5s$ and is then removed. Give the differential equation with initial conditions for this motion by using the step function $u_c(t)$. Do **NOT** solve this differential equation.

$m = 1,\quad k = 13,\quad \gamma = 6.$

3

$$y'' + 6y' + 13y = 3(u_2(t) - u_5(t)),\quad y(0) = 0,\quad y'(0) = -1.$$ 2.

(b) (15 points) Suppose an impulsive external force $\delta(t-2)$ is applied to the system. Find the position of the mass at any time t using **Laplace transforms**. No credit will be given for any other methods.

$$y'' + 6y' + 13y = \delta(t-2),\quad y(0) = 0,\quad y'(0) = -1.$$ 3

$\mathcal{L}\{y(t)\} = Y(s),\quad \mathcal{L}\{y'\} = sY(s).\quad \mathcal{L}\{y''\} = s^2Y(s) + 1$

$$s^2Y(s) + 1 + 6sY(s) + 13Y(s) = e^{-2s}$$

$$(s^2 + 6s + 13)Y(s) = e^{-2s} - 1.$$ 5

$$Y(s) = e^{-2s}\frac{1}{s^2+6s+13} - \frac{1}{s^2+6s+13}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+4}\right\} = \frac{1}{2}e^{-3t}\sin 2t.$$ 3 + 3.

$$y(t) = u_2(t)\frac{1}{2}e^{-3(t-2)}\sin 2(t-2) - \frac{1}{2}e^{-3t}\sin 2t.$$

1

15. (15 points)

(a) (8 points) Find the general solution of the system:

$$\begin{cases} x_1' = x_2 \\ x_2' = 2x_1 + x_2 \end{cases} \qquad \vec{X}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \vec{X}$$

$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

I. Find eigenvalues of A: char poly of $A = \det(A - rI) = \begin{vmatrix} -r & 1 \\ 2 & 1-r \end{vmatrix}$

$= -r(1-r) - 2 = r^2 - r - 2$

$= (r-2)(r+1)$

2 Eigenvalues are $2, -1$

II. Find a basis of 2-eigenspace.

2 Solve $A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, i.e. $(A - 2I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e. $-2v_1 + v_2 = 0$, $v_2 = 2v_1$ ~~basis~~.

2-eigenspace $= \left\{\begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} : v_1 \text{ arb}\right\}$ has a basis $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2 III. Find a basis of -1-eigenspace.

Solve $A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, i.e. $(A + I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $v_1 + v_2 =$

-1-eigenspace $= \left\{\begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} : v_1 \text{ arb}\right\}$ has a basis $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

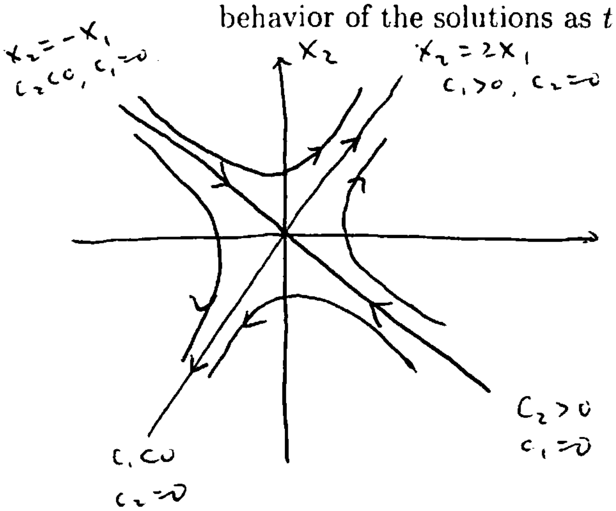
IV. $\vec{Z}_1 = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$, $\vec{Z}_2 = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$

2 Gen'l sol'n $\vec{X} = c_1\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + c_2\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$, c_1, c_2 consts.

(b) (2 points) Discuss the **type** and the **stability** of the critical point $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Type: saddle pt. unstable.

(c) (5 points) Sketch the phase portrait of the given system. Clearly indicate the behavior of the solutions as $t \to \infty$.



$\vec{Z}_1$ $x_2 = 2x_1$ $\vec{Z}_2$: $x_2 = -x_1$

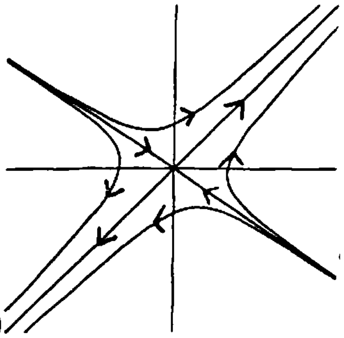
1 1

16. (21 points) Classify the type of the critical point at the origin, its stability and choose a phase portrait (**given on the next page**) of the system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, where the matrix $\boldsymbol{A}$ has the following eigenvalues r_1, r_2.

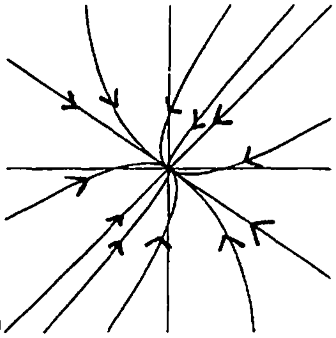
	EIGENVALUES	TYPE	STABILITY	PHASE PORTRAIT
(a)	$r_1 = r_2 = -1$ with two linearly independent eigenvectors;	proper node	asymp. stable	G
(b)	$r_1 = -2, r_2 = 3$;	saddle pt	unstable	A
(c)	$r_1 = 2, r_2 = 4$;	source node	unstable	C
(d)	$r_1 = 2i, r_2 = -2i$;	center	stable	F
(e)	$r_1 = -1, r_2 = -5$;	sink node	asymp stable	B
(f)	$r_1 = r_2 = 2$ with only one linearly independent eigenvector;	improper node	unstable	E
(g)	$r_1 = 1 + i, r_2 = 1 - i$.	spiral pt	unstable	I

Types of the critical point are: saddle point, node, center, spiral point, proper node, and improper node.

A critical point may be: stable, asymptotically stable, or unstable.



(A)



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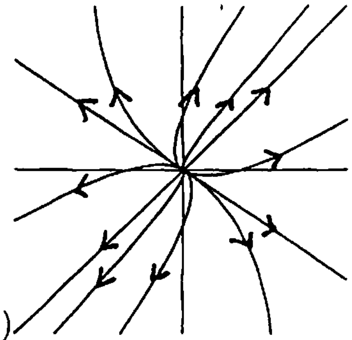
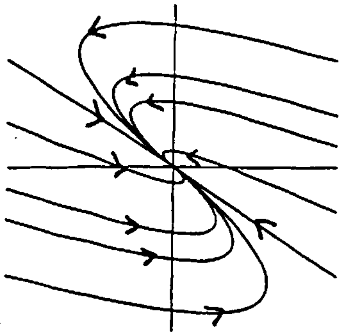


(C)



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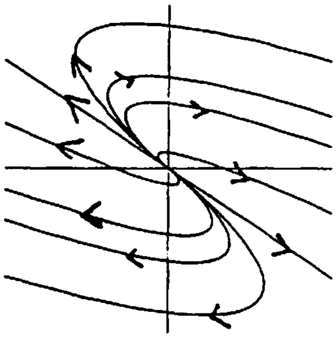

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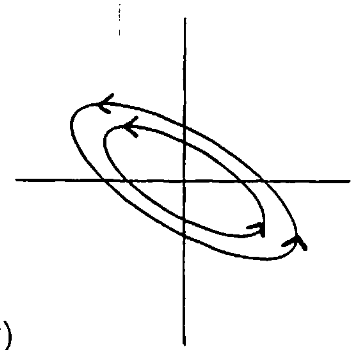

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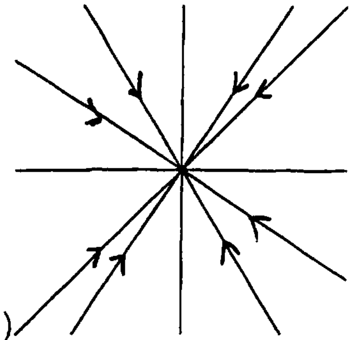

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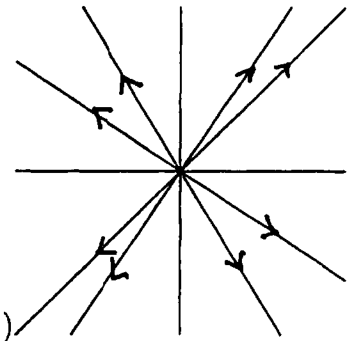

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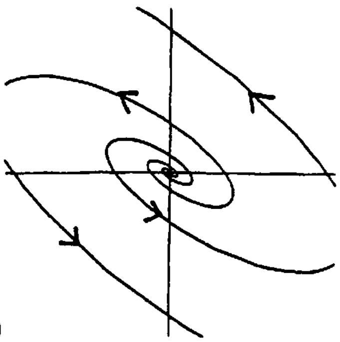
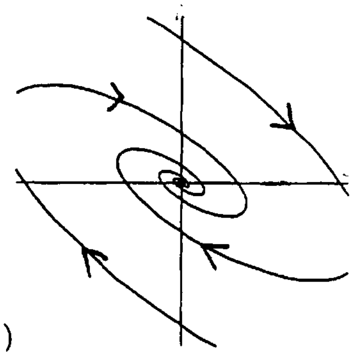


(I)



(J)

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n; n = positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, n = positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28