Math 250
Fall 2007
Exam 1

NAME: ______________________

ID No: ______________________

SECTION: ______________________

This exam contains 10 questions on 9 pages (including this title page). This exam is worth a total of 100 points. The exam is broken into two parts. There are six multiple choice questions, each worth 5 points, and 4 partial credit problems. To receive full credit for a partial credit problem all work must be shown. When in doubt, fill in the details.

No notes, books or calculators may be used during the exam.

Please, Box Your Final Answer (when possible).

1:	B
2:	C
3:	C
4:	A
5:	A
6:	B
7:	
8:	
9:	
10:	
Total:	

Multiple Choice Section

1. *(5 points)* Determine the order and linearity of the differential equation:

$$(y')^2 + \sin(t)y' - y = t^3.$$

(a) First order and linear.

✓ (b) First order and nonlinear.

(c) Second order and linear.

(d) Second order and nonlinear.

2. *(5 points)* Which of the following will be an integrating factor for the differential equation

$$ty' + (t+2)y = t^3?$$

DO NOT attempt to solve this differential equation.

$$y' + \frac{t+2}{t} y = t^2$$

$$e^{\int \frac{t+2}{t} dt} = e^{\int 1 + \frac{2}{t} dt} = e^{t + 2 \ln t}$$

$$= t^2 e^t$$

(a) $e^{\frac{t^2}{2}} + e^{2t}$.

(b) $e^{\frac{t^2}{2}+2t}$.

✓ (c) t^2e^t.

(d) $t^2 + e^t$.

3. *(5 points)* Find the maximal interval where the existence and uniqueness of the solution to the initial value problem is guaranteed:

$$\ln(t)y' + \tan(t)y = \sin^2(t), \qquad y\left(\frac{\pi}{3}\right) = -1.$$

(Note that $\pi \approx 3.14$)

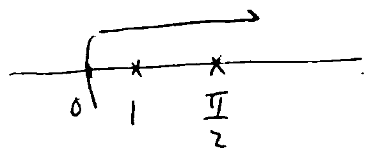
$$y' + \frac{\tan t}{\ln t} y = \frac{\sin^2 t}{\ln t}$$

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\left(0, \frac{\pi}{2}\right)$.

(c) $\left(1, \frac{\pi}{2}\right)$.

(d) $\left(-\frac{\pi}{2}, 0\right)$.



4. *(5 points)* You win one million dollars in the lottery. You plan to invest the money in a bank at 5% annual interest rate. At the same time you will withdraw 50 thousand dollars every year to travel. Which of the following best describes the amount of money $S(t)$ you will have after t years?

Assume that the interest is compounded continuously and money is withdrawn continuously as well.

(a) $\frac{d}{dt}S(t) = .05S(t) - 50,000$, $S(0) = 1,000,000$.

(b) $\frac{d}{dt}S(t) = .05S(t) + 50,000$, $S(0) = 1,000,000$.

(c) $\frac{d}{dt}S(t) = 5S(t) + 1,000,000$, $S(0) = 1,000,000$.

(d) $S(t) = (.05)^t 1,000,000 - 50,000t$.

5. *(5 points)* Suppose that

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = te^t$$

constitute a fundamental set of solutions to a homogeneous second order linear differential equation.

Find the pair which DOES NOT constitute a fundamental set of solutions to the same differential equation.

(a) $y_3(t) = e^{t+3}, \qquad y_4(t) = e^{t-2}.$

(b) $y_3(t) = e^t, \qquad y_4(t) = (t+1)e^t.$

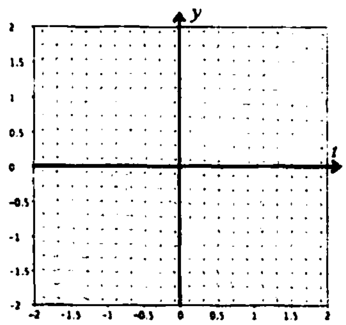
(c) $y_3(t) = e^{t+1}, \qquad y_4(t) = (t+1)e^{t+1}.$

(d) $y_3(t) = 2e^t, \qquad y_4(t) = -te^t.$

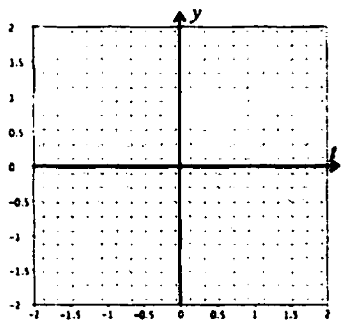
6. *(5 points)* Which of the following is the direction field for the differential equation

$$y' = y^2 + t?$$

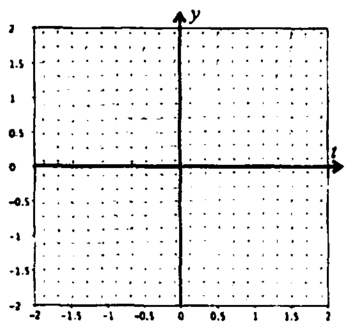
(a)



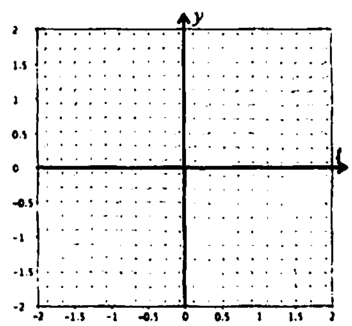
(c)



(b)



(d)



Partial Credit Section

7. For the initial value problem

$$y' = \frac{3x^2 + 2x + 1}{2y - 4}, \qquad y(0) = 1$$

(a) *(10 points)* Solve for $y(x)$ explicitly in terms of x.

$\int 2y - 4 \, dy = \int 3x^2 + 2x + 1 \, dx$ 2 pt

$y^2 - 4y = x^3 + x^2 + x + C$ 1

$y(0) = 1 \Rightarrow 1 - 4 = C = -3.$ 2

$y^2 - 4y + 4 = x^3 + x^2 + x - 3 + 4$ 3

$(y-2)^2 = x^3 + x^2 + x + 1$

$y - 2 = \pm\sqrt{x^3 + x^2 + x + 1}$

$y = 2 \pm \sqrt{x^3 + x^2 + x + 1}$

$y(0) = 2 \pm 1 = \{3, 1\}$

Choose $-$ 2 pts.

Ans. $y = 2 - \sqrt{x^3 + x^2 + x + 1}$.

(b) *(4 points)* Find the maximal interval (a, b) on which the above solution is valid. *(If it is difficult to find a and b, you may describe them as zeros of a certain function).*

Want $x^3 + x^2 + x + 1 = x(x^2+1) + x^2 + 1 = (x+1)(x^2+1) > 0$. i.e. $x > -1$. 3 pts

Interval $(-1, \infty)$. 1

(c) *(4 points)* Find the limits of $y(x)$ as x approaches the end points of the interval (a, b) from part **(b)**.

$\lim_{x \to \infty} y = -\infty$, 2

$\lim_{x \to -1^+} y = 2$. 2

8. A tank initially contains 100 liters of fresh water. A mixture containing 10 grams of salt per liter is poured into the tank at the rate of 2 liters per minute. The well-stirred mixture is allowed to leave the tank at the same rate.

(a) *(13 points)* Set up the initial value problem and find the amount of salt in the tank after 10 minutes.

In. 10 g/l, 2 l/min

2 l/min

out $\frac{Q(t)}{100}$ g/l

$Q(t)$ = amount of salt in g after t min.

$$\frac{dQ}{dt} = 10 \times 2 - 2 \cdot \frac{Q(t)}{100}, \quad Q(0) = 0. \quad 5 \text{ pts}$$

$$Q' + \frac{1}{50} Q = 20, \quad Q(0) = 0.$$

$$\mu(t) = e^{\int \frac{1}{50} dt} = e^{\frac{1}{50}t} \quad 2 \text{ pts} \quad \left(e^{\frac{1}{50}t} Q\right)' = 20 e^{\frac{1}{50}t}.$$

$$e^{\frac{1}{50}t} Q(t) = 20 \frac{e^{\frac{1}{50}t}}{\frac{1}{50}} + c = 1000 e^{\frac{1}{50}t} + c$$

$$\underline{Q(t) = 1000 + c e^{-\frac{1}{50}t}} \quad 3 \text{ pts} \qquad Q(0) = 1000 + c = 0. \quad c = -1000. \quad 2 \text{ pts}$$

$$Q(t) = 1000 - 1000 e^{-\frac{1}{50}t} \qquad Q(10) = 1000 - 1000 e^{-\frac{1}{5}} \text{ g}. \quad 1$$

(b) *(6 points)* Find the time T when the concentration of salt in the tank is 5 grams per liter.

4

$$\text{Concentration} = \frac{Q(T)}{100} = 10 - 10 e^{-\frac{1}{50}T} = 5. \quad 3 \text{ pts for settip up}$$

$$5 = 10 e^{-\frac{1}{50}T} \qquad \frac{1}{2} = e^{-\frac{1}{50}T}.$$

3 pts solving

$$-\frac{1}{50}T = \ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2.$$

$$T = 50 \ln 2 \text{ min}.$$

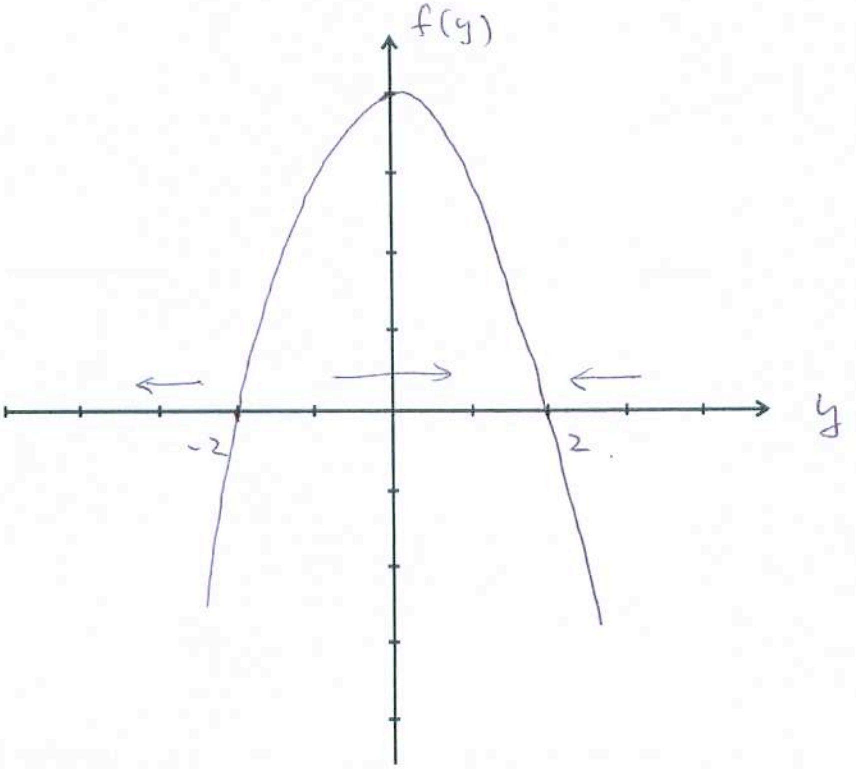
(c) *(1 points)* What is the limit concentration of salt?

$$\lim_{t \to \infty} \frac{Q(t)}{100} = 10 \text{ g/l}. \qquad 1 \text{ pt}.$$

9. Consider the autonomous equation

$$\frac{dy}{dt} = 4 - y^2.$$

(a) *(4 points)* Sketch the graph of $f(y) = 4 - y^2$. $= (2-y)(2+y)$



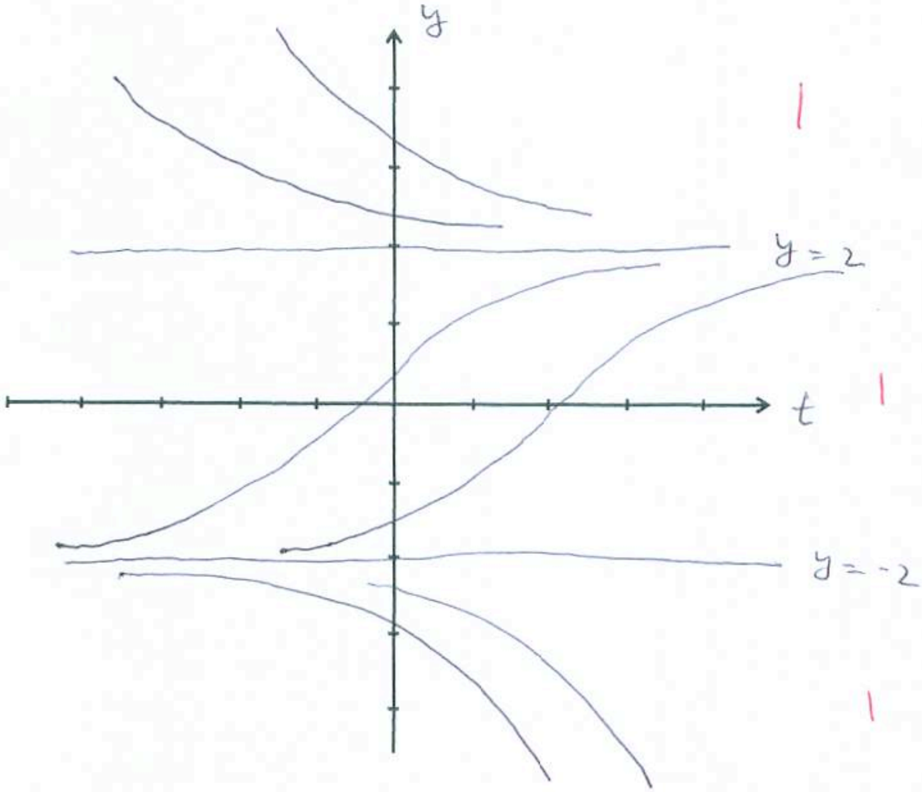
(b) *(6 points)* Find and classify the equilibrium solutions.

Two equilib. sol'ns

$y = -2$ unstable 3

$y = 2$ asymp. stable 3

(c) *(4 points)* Sketch the graphs of several solutions, making sure you have at least one graph representing each type.



(d) *(3 points)* Without solving the equation, find the limit as $t \to \infty$ of the solution satisfying the initial condition $y(2) = -5$.

Since $y_0 = -5$. $y \to -\infty$ as $t \to \infty$.

10. For the second order differential equation

$$y'' - 2y' - 3y = 0$$

(a) *(10 points)* Find the general solution $y(t)$.

Char eq'n $r^2 - 2r - 3 = 0$ $(r-3)(r+1) = 0$. 3

$r_1 = 3, \ r_2 = -1$ 2

Get two sol'ns $y_1 = e^{3t}, \quad y_2 = e^{-t}$

Check $W(y_1, y_2) = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} = -e^{3t}e^{-t} - 3e^{3t}e^{-t} = -4e^{3t}e^{-t} \neq 0$

$\therefore \{y_1, y_2\}$ is a fund. set of sol'ns.

5

Gen'l sol'n $y = c_1 e^{3t} + c_2 e^{-t}$, c_1, c_2 any consts.

(b) *(5 points)* Find the value α, such that the solution in **(a)** satisfying the initial condition

$$y(0) = \alpha, \qquad y'(0) = 1$$

remains finite as $t \to \infty$.

In order that sol'n remains finite as $t \to \infty$, $c_1 = 0$.

$\therefore y = c_2 e^{-t}$

$y' = -c_2 e^{-t}$

$y'(0) = -c_2 = 1, \quad c_2 = -1$

$y = -e^{-t}$

$y(0) = -1 = \alpha$

Knowing $c_1 = 0$ 2 pts

Another way

$y = c_1 e^{3t} + c_2 e^{-t}$

$y' = 3c_1 e^{3t} - c_2 e^{-t}$

$y(0) = c_1 + c_2 = \alpha$ ① 1

$y'(0) = 3c_1 - c_2 = 1$ ② 1

① + ② $4c_1 = \alpha + 1$ $c_1 = \frac{\alpha+1}{4}$.

$c_2 = \alpha - c_1 = \alpha - \frac{\alpha+1}{4} = \frac{3\alpha - 1}{4}$

$y = \frac{\alpha+1}{4} e^{3t} + \frac{3\alpha-1}{4} e^{-t}$ 1

y remains bdd $\Rightarrow \frac{\alpha+1}{4} = 0$, i.e. $\alpha = -1$.