Multiple Choice Section

1. (5 points) Which of the following ordinary differential equations is linear of the second order?
   (a) $y'' + 4yy' = 0$
   (b) $t^2y'' + ty' - y = \ln t$
   (c) $(y')^2 + 2y = 0$
   (d) $t^2 + y' = e^{-t}$

2. (5 points) Which of the following will be an integrating factor for the differential equation $ty' - 2y = 2\cos 2t$?

- (a) $\frac{1}{t^2}$
- (b) $e^{-2t}$
- (c) $-t^2$
- (d) $-2t$

\[
y' - \frac{2}{t} y = \frac{2e^{2t}}{t} \]

\[
\int -\frac{2}{t} \, dt = \ln|t| = t^{-2}
\]
3. (5 points) Find the maximal interval in which the existence and uniqueness of the solution to the following initial value problem is guaranteed:

\[(\sin t)y' + \frac{1}{t-1}y = \frac{t}{t+1}, \quad y(2) = 3.\]

(a) \((0, \pi)\)
(b) \((0, 2)\)
(c) \((-1, \pi)\)
(d) \((1, \pi)\)

\[y' + \frac{1}{(t-1)(t-1)}y = \frac{t}{(t+1) \sin t}\]

\[\begin{array}{cccc}-\pi & -1 & 0 & 1 & \pi & 2\pi \\
\end{array}\]

4. (5 points) Find a second order linear differential equation with constant coefficients to which the following functions are solutions:

\[y_1(t) = e^{-2t}, \quad y_2(t) = e^t.\]

(a) \(y'' - y' - 2y = 0\)
(b) \(y'' + 2y' - 3y = 0\)
(c) \(y'' + y' - 2y = 0\)
(d) \(y'' + y' + 2y = 0\)

\(r_1 = -2, \quad r_2 = 1\)
\((r+2)(r-1) = 0\)
\(r^2 + r - 2 = 0\)
\(y'' + y' - 2y = 0\)
5. (5 points) Which of the following is the direction field for the differential equation \( y' = y - t \)?

(a) \[ \begin{array}{c}
\end{array} \]

(b) \[ \begin{array}{c}
\end{array} \]

6. (5 points) Consider the following two pairs of solutions:

\[
\begin{align*}
y_1(t) &= e^t \\
z_1(t) &= \sin t \\
y_2(t) &= e^{t+1} \\
z_2(t) &= \sin(2t)
\end{align*}
\]

Which of the following is true?

(a) Both of these pairs form fundamental sets of solutions.
(b) Only the second pair forms a fundamental set of solutions.
(c) Only the first pair forms a fundamental set of solutions.
(d) Neither pairs form fundamental sets of solutions.
Partial Credit Section

7. (a) (10 points) Solve the initial value problem

\[ y' = \frac{x^3 + 1}{yx^2 - x^2}, \quad y(1) = 0. \]

Write your answer \( y \) as an explicit function of \( x \).

\[
\frac{dy}{dx} = \frac{x^3 + 1}{(y-1)x^2}, \quad y(1) = 0
\]
\[
\int (y-1) \, dy = \int \frac{x^3 + 1}{x^2} \, dx = \int x + \frac{1}{x} \, dx
\]
\[
\frac{y^2}{2} - y = \frac{x^2}{2} - \frac{1}{x} + C_1
\]
\[
\frac{1}{2}(y-1)^2 = \frac{x^2}{2} - \frac{1}{x} + C
\]
\[
\frac{y-1}{x} = \frac{x^2}{x} - \frac{1}{x} + C_2
\]
\[
y = 1 - \frac{2}{x} + C_2
\]
\[
y(1) = 0 \implies 1 = 1 - 2 + C_2 \implies C_2 = 2
\]
\[
y = 1 = 1 + \sqrt{x^2 - \frac{2}{x} + 2}
\]
\[
y(1) = 0 \implies \text{choose sign.} \quad y = 1 - \sqrt{x^2 - \frac{2}{x} + 2}
\]

(b) (5 points) Find the maximal possible interval \((a, b)\) on which the above solution is valid. If it is difficult to solve for \( a \) or \( b \) without a calculator, you may describe it as a zero of a certain function.

Vertical tangent at \( y = 0 \), \( x^2 - \frac{2}{x} + 2 = 0 \). Want \((a, b)\) containing 1 on which \( f > 0 \).

Let \( f(x) = x^2 - \frac{2}{x} + 2 \). \( f(1) = 1 - 2 + 2 = 1 > 0 \).

As \( x \to \infty \), \( f(x) = 2x + \frac{2}{x^2} > 0 \). \( m(0, \infty) \).

As \( x \to 0^+ \), \( f(x) \to -\infty \). Let \( a \) be the \( \text{positive} \) zero of \( f \), which \( a > 0 \).

It exists by intermediate value theorem. (Only one such since \( f \) is increasing.)

Ans. \((a, \infty)\), where \( a \) is the positive zero of \( f(x) = x^2 - \frac{2}{x} + 2 \).
8. (20 points) A tank initially contains 60 gal of pure water. Sweet water containing 1 lb of sugar per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at the same rate.

(a) (13 points) Set up and solve an initial value problem for the amount of sugar $Q$ at any time $t$.

\[ \frac{dQ}{dt} = 1 \cdot 2 - \frac{Q}{60} \cdot 2 = 2 - \frac{1}{30} Q , \quad Q(0) = 0 \]

Solving, we get

\[ Q(t) = 60 + Ce^{-\frac{1}{30} t} \]

Using $Q(0) = 0$,

\[ 0 = 60 + C \Rightarrow C = -60 \]

\[ Q(t) = 60 - 60e^{-\frac{1}{30} t} \]

(b) (7 points) Find an instant of time $T$ when the concentration of the sugar is 1/2 lb per gal.

Concentration = \( \frac{Q(t)}{60} = 1 - e^{-\frac{1}{30} t} = \frac{1}{2} \)

\[ e^{-\frac{1}{30} t} = \frac{1}{2} \]

\[ -\frac{1}{30} t = \ln \frac{1}{2} = -\ln 2 \]

\[ T = 30 \cdot \ln 2 \text{ min} \]
9. (20 points) Consider the following autonomous equation:

\[ y' = y^2 - 4y. \]

(a) (3 points) Sketch the graph of \( f(y) = y^2 - 4y = y(y - 4) \)

(b) (6 points) Find the critical points and classify equilibrium solutions for this differential equation.

Critical pts \( y = 0, 4 \quad \text{2} \)

Equilib. soln. \( y = 0 \) asymp. stable \( \quad \text{2} \)
\( y = 4 \) unstable \( \quad \text{2} \)

c) (3 points) Find inflection points.

\[ f''(y) = 2y - 4, \quad \text{Inflection pt} \quad \text{at} \quad y = 2 \quad \text{3} \]
Question 9 continued.
(d) (4 points) Sketch the graphs of several solutions, making sure you have at least one graph representing each type.

(e) (4 points) Find the limit as $t \to \infty$ of the solution satisfying the initial condition $y(1) = 3$, without solving the equation.

$$\lim_{y \to \infty} y = 0$$
10. (a) (10 points) Find the general solution to the second order differential equation

\[ y'' + y' - 2y = 0. \]

Char eqn \[ r^2 + r - 2 = 0 \quad \Rightarrow \quad (r + 2)(r - 1) = 0 \]

\[ r = -2, 1 \]

\[ y_1 = e^{-2t} \quad y_2 = e^t \]

Gen soln \[ y = c_1 e^{-2t} + c_2 e^t \]

(b) (5 points) Find the value \( \beta \) such that the solution in (a) satisfying the initial conditions

\[ y(0) = 3, \quad y'(0) = \beta \]

remains finite as the independent variable \( t \to \infty \).

Since \( e^t \to \infty \) and \( e^{-2t} \to 0 \) as \( t \to \infty \), in order that \( y \) remains bounded, we must have \( c_2 = 0 \). So

\[ y = c_1 e^{-2t} \]

\[ y(0) = 3 \quad \Rightarrow \quad c_1 = 3 \]

\[ y' = -6 e^{-2t} \quad y'(0) = -6 = \beta \]