Multiple Choice Section

1. (5 points) Which of the following ordinary differential equations is linear of the second order?

   (a) $y'' + 4yy' = 0$
   (b) $t^2y'' + ty' - y = \ln t$
   (c) $(y')^2 + 2y = 0$
   (d) $t^2 + y' = e^{-t}$

2. (5 points) Which of the following will be an integrating factor for the differential equation

$$ty' - 2y = 2\cos 2t?$$

✓(a) $\frac{1}{t^2}$
(b) e^{-2t}
(c) $-t^2$
(d) $-2t$

$$y' - \frac{2}{t}y = \frac{2\cos 2t}{t}$$

$$e^{\int -\frac{2}{t}\,dt} = e^{-2\ln|t|} = t^{-2}$$

3. (5 points) Find the maximal interval in which the existence and uniqueness of the solution to the following initial value problem is guaranteed:

$$(\sin t)y' + \frac{1}{t-1}y = \frac{t}{t+1}, \qquad y(2) = 3.$$

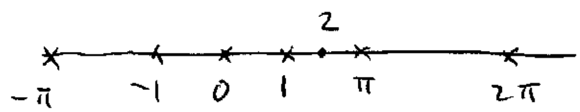
(a) $(0, \pi)$

(b) $(0, 2)$

(c) $(-1, \pi)$

(d) $(1, \pi)$

$$y' + \frac{1}{(\sin t)(t-1)} y = \frac{t}{(t+1)\sin t}$$



4. (5 points) Find a second order linear differential equation with constant coefficients to which the following functions are solutions:

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{t}.$$

(a) $y'' - y' - 2y = 0$

(b) $y'' + 2y' - 3y = 0$

(c) $y'' + y' - 2y = 0$

(d) $y'' + y' + 2y = 0$

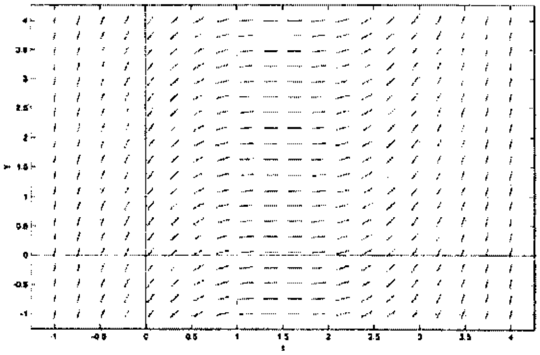
$r_1 = -2 \qquad r_2 = 1$

$(r+2)(r-1) = 0$

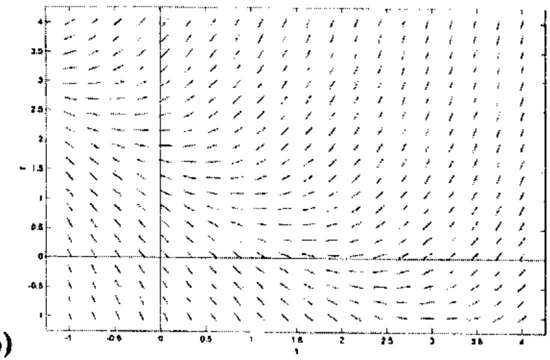
$r^2 + r - 2 = 0$

$y'' + y' - 2y = 0$

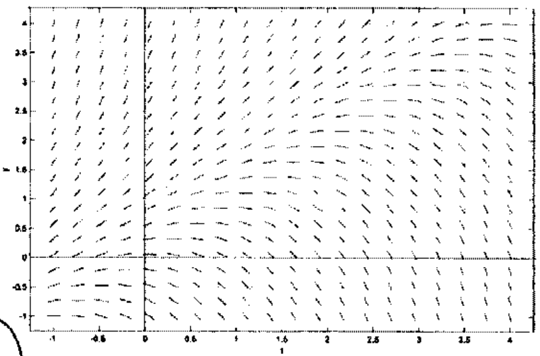
5. (5 points) Which of the following is the direction field for the differential equation $y' = y - t$?



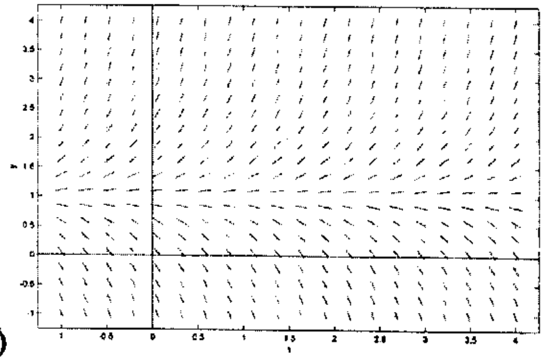
(a)



(b)



~~**(a)**~~ c



~~**(b)**~~ d

6. (5 points) Consider the following two pairs of solutions:

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = e^{t+1}$$
$$z_1(t) = \sin t \qquad \text{and} \qquad z_2(t) = \sin(2t)$$

Which of the following is true?

(a) Both of these pairs form fundamental sets of solutions.

(b) Only the second pair forms a fundamental set of solutions.

(c) Only the first pair forms a fundamental set of solutions.

(d) Neither pairs form fundamental sets of solutions.

Answer key #1 B #2 A #3 D #4 C #5 C #6 B

Partial Credit Section

7. (a) (10 points) Solve the initial value problem

$$y' = \frac{x^3+1}{yx^2 - x^2}, \qquad y(1) = 0.$$

Write your answer y as an explicit function of x.

$\frac{dy}{dx} = \frac{x^3+1}{(y-1)x^2}$, $y(1)=0$

$\int (y-1)\,dy = \int \frac{x^3+1}{x^2}\,dx = \int x + \frac{1}{x^2}\,dx$

$\frac{1}{2}(y-1)^2 = \frac{x^2}{2} - \frac{1}{x} + c$

$(y-1)^2 = x^2 - \frac{2}{x} + c_2$. 5

$\frac{y^2}{2} - y = \frac{x^2}{2} - \frac{1}{x} + c_1$

$y^2 - 2y = x^2 - \frac{2}{x} + c_2$

$(y-1)^2 = x^2 - \frac{2}{x} + c_3$

$y(1)=0 \Rightarrow 1 = 1 - 2 + c_2 \quad c_2 = 2$ 2

$y - 1 = \pm\sqrt{x^2 - \frac{2}{x} + 2}$, or $y = 1 \pm \sqrt{x^2 - \frac{2}{x} + 2}$

$y(1)=0 \Rightarrow$ chose − sign. $y = 1 - \sqrt{x^2 - \frac{2}{x} + 2}$. 3

(b) (5 points) Find the maximal possible interval (a, b) on which the above solution is valid. If it is difficult to solve for a or b without a calculator, you may describe it as a zero of a certain function.

Vertical tangent at $y=1$, so $x^2 - \frac{2}{x} + 2 = 0$. Want (a,b) containing 1 (on which $f > 0$.

Let $f(x) = x^2 - \frac{2}{x} + 2$ $\quad f(1) = 1 - 2 + 2 = 1 > 0$.

As $x \to \infty$, $f(x) \to \infty$ $\quad f'(x) = 2x + \frac{2}{x^2} > 0$ on $(0,\infty)$. $\therefore f \uparrow$ on $(0,\infty)$

As $x \to 0^+$, $f(x) \to -\infty$, let a be the ~~largest~~ zero of f, which is >0 ~~which is >0~~.

It exists by intermediate value theorem. (Only one such since f is increasing.)

Ans. (a, ∞), where a is the positive zero of $f(x) = x^2 - \frac{2}{x} + 2$.

8. (20 points) A tank initially contains 60 gal of pure water. Sweet water containing 1 lb of sugar per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at the same rate.

(a) (13 points) Set up and solve an initial value problem for the amount of sugar Q at any time t.

1 lb/gal
2 gal/min

→ 2 gal/min.

$$\frac{dQ}{dt} = 1\cdot 2 - \frac{Q}{60}\cdot 2 = 2 - \frac{1}{30}Q, \quad Q(0) = 0$$ 6 pts

$$Q' + \frac{1}{30}Q = 2, \quad Q(0) = 0.$$

integrating factor $e^{\int \frac{1}{30}dt} = e^{\frac{1}{30}t}$ 2 pts

$$\left(e^{\frac{1}{30}t}Q\right)' = 2e^{\frac{1}{30}t}. \qquad e^{\frac{1}{30}t}Q(t) = \int 2e^{\frac{1}{30}t}\,dt = 2\cdot 30\,e^{\frac{1}{30}t} + c$$

$$Q(t) = 60 + c\,e^{-\frac{1}{30}t}$$ 3 pts

$$Q(0) = 0 \Rightarrow \quad 0 = 60 + c, \quad c = -60$$ 2 pts

$$Q(t) = 60 - 60e^{-\frac{1}{30}t}.$$

(b) (7 points) Find an instant of time T when the concentration of the sugar is 1/2 lb per gal.

$$\text{Concentration} = \frac{Q(t)}{60} = 1 - e^{-\frac{1}{30}t} = \frac{1}{2}$$

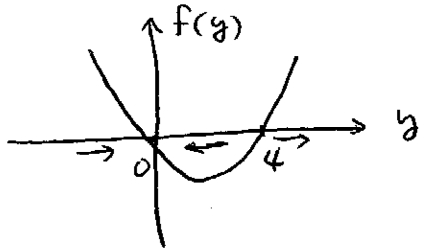
$$e^{-\frac{1}{30}t} = \frac{1}{2}$$

$$-\frac{1}{30}t = \ln\frac{1}{2} = -\ln 2. \qquad T = 30\cdot \ln 2 \text{ min.}$$

9. (20 points) Consider the following autonomous equation:

$$y' = y^2 - 4y.$$

(a) (3 points) Sketch the graph of $f(y) = y^2 - 4y$. $= y(y-4)$



(b) (6 points) Find the critical points and classify equilibrium solutions for this differential equation.

Critical pts $y = 0, 4$ 2

Equilib. soln $y = 0$ asymp. stable 2

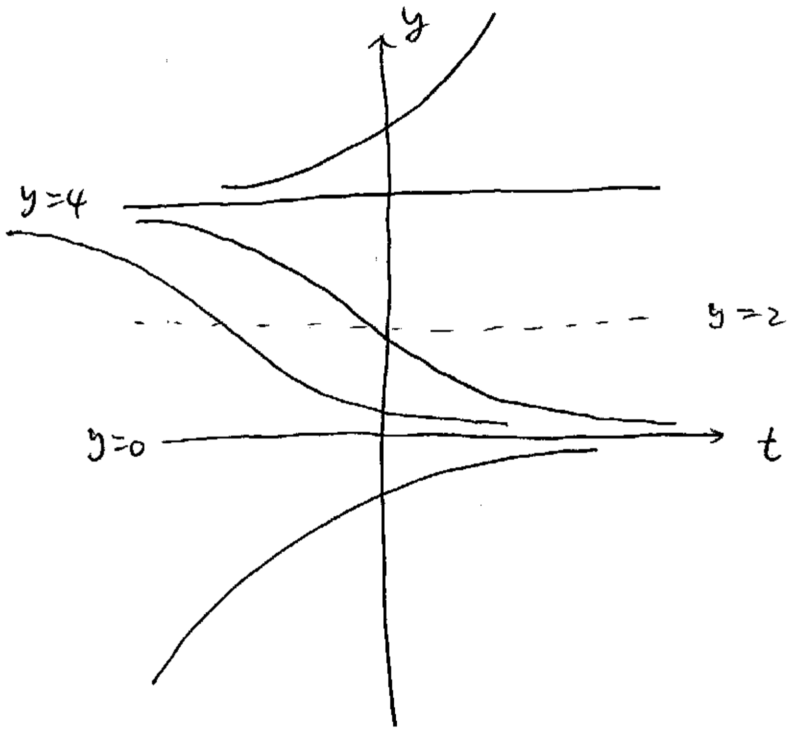
$y = 4$ unstable 2

(c) (3 points) Find inflection points.

$f'(y) = 2y - 4$, Inflection pts on $y = 2$. 3

Question 9 continued.

(d) (4 points) Sketch the graphs of several solutions, making sure you have at least one graph representing each type.



(e) (4 points) Find the limit as $t \to \infty$ of the solution satisfying the initial condition $y(1) = 3$, without solving the equation.

$\lim_{y \to \infty} y = 0.$

10. (a) (10 points) Find the general solution to the second order differential equation

$$y'' + y' - 2y = 0.$$

1

Char eq'n $r^2 + r - 2 = 0$. $(r+2)(r-1) = 0$

$r = -2, 1$ 3

$y_1 = e^{-2t}$, $y_2 = e^{t}$ 2 + 2

Gen'l sol'n $y = c_1 e^{-2t} + c_2 e^{t}$. 1

(b) (5 points) Find the value β such that the solution in (a) satisfying the initial conditions

$$y(0) = 3, \qquad y'(0) = \beta$$

remains finite as the independent variable $t \to \infty$.

Since $e^t \to \infty$ and $e^{-2t} \to 0$ as $t \to \infty$, in order that y remains bounded, we must have $c_2 = 0$. So

$y = c_1 e^{-2t}$. 3

$y(0) = 3 \Rightarrow c_1 = 3$. $\therefore$ $y = 3e^{-2t}$. 1

$y'(t) = -6e^{-2t}$. $y'(0) = -6 = \beta$. 1